Nov. 28 Math 1190 sec. 51 Fall 2016

Section 5.4: Properties of the Definite Integral

Suppose that f and g are integable on $[a, b]$ and let k be constant.

I. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

II. $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

II. $\displaystyle\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

The Sum/Difference in General

If $f_1, f_2, \ldots, f_n$ are integrable on $[a, b]$ and $k_1, k_2, \ldots, k_n$ are constants, then

$$\int_a^b [k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x)]\, dx =$$

$$k_1 \int_a^b f_1(x)\, dx + k_2 \int_a^b f_2(x)\, dx + \cdots + k_n \int_a^b f_n(x)\, dx$$

Properties of Definite Integrals Continued...

If f is integrable on any interval containing the numbers a, b, and c, then

(IV) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example

Suppose $F'(x) = f(x)$ for all x. Show that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

LHS RHS

LHS $\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$

RHS $\int_a^c f(x)dx + \int_c^b f(x)dx = F(x)\Big|_a^c + F(x)\Big|_c^b$

$= F(c) - F(a) + F(b) - F(c) = F(b) - F(a)$

Example

Suppose

$$\int_{-1}^{4} f(x)\,dx = -2, \quad \text{and} \quad \int_{2}^{4} f(x)\,dx = 3.$$

Evaluate $\displaystyle\int_{-1}^{2} f(x)\,dx$

$$\int_{-1}^{4} f(x)\,dx = \int_{-1}^{2} f(x)\,dx + \int_{2}^{4} f(x)\,dx$$

$$-2 = \int_{-1}^{2} f(x)\,dx + 3$$

$$\Rightarrow \int_{-1}^{2} f(x)\,dx = -2 - 3 = -5$$

Question

Suppose

$$\int_0^5 f(x)\,dx = 3, \quad \text{and} \quad \int_0^7 f(x)\,dx = 1.$$

Evaluate $\displaystyle\int_5^7 f(x)\,dx$

$$\int_0^7 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx$$

$$1 = 3 + \int_5^7 f(x)\,dx$$

(a) 2

(b) -2

(c) 4

(d) Can't be determined without more information.

Properties: Bounds on Integrals

(V) If $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

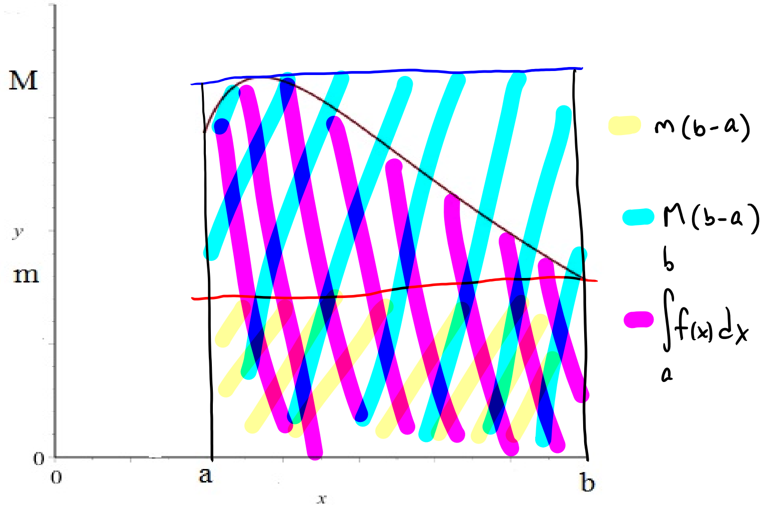
If $f(x) \geq 0$ on $[a,b]$ then $\displaystyle\int_a^b f(x)\,dx \geq 0$

(VI) And, as an immediate consequence of (V), if $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

If f is continuous on $[a, b]$, we can take m to be the absolute minimum value and M the absolute maximum value of f as guaranteed by the Extreme Value Theorem.

Bounding: For nonnegative function



Average Value of a Function and the Mean Value Theorem

Defintion: Let f be continuous on the closed interval $[a, b]$. Then the average value of f on $[a, b]$ is

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Theorem: (The Mean Value Theorem for Integrals) If f is continuous on the interval $[a, b]$, then there exists a number u in $[a, b]$ such that

$$f(u) = f_{avg}, \quad \text{i.e.} \quad \int_a^b f(x)\,dx = f(u)(b-a).$$

Question

Find the average value of $f(x) = \sqrt{x}$ on the interval $[0, 4]$. That is, compute

$$f_{avg} = \frac{1}{4-0} \int_0^4 x^{1/2}\, dx$$

$$= \frac{1}{4} \left[\frac{x^{3/2}}{3/2} \right|_0^4 = \frac{1}{4}\left(\frac{2}{3}\right)\left(4^{3/2} - 0^{3/2}\right)$$

$$= \frac{1}{2 \cdot 3}\left(8 - 0\right) = \frac{8}{6} = \frac{4}{3}$$

(a) $\frac{16}{3}$

(b) $\frac{4}{3}$

(c) $\frac{1}{2}$

(d) 2

Question

Find the value of f guaranteed by the MVT for integrals for $f(x) = \sqrt{x}$ on the interval $[0, 4]$. That is, find u such that

$$f(u) = f_{avg} = \frac{1}{4}\int_0^4 x^{1/2}\,dx = \frac{4}{3}$$

(a) $\sqrt{\frac{4}{3}}$

(b) $\frac{2}{\sqrt{3}}$

(c) $\frac{16}{9}$

(d) $\frac{16}{3}$

$f(u) = \frac{4}{3}$

$\sqrt{u} = \frac{4}{3} \Rightarrow u = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$

MVT for Integrals Example

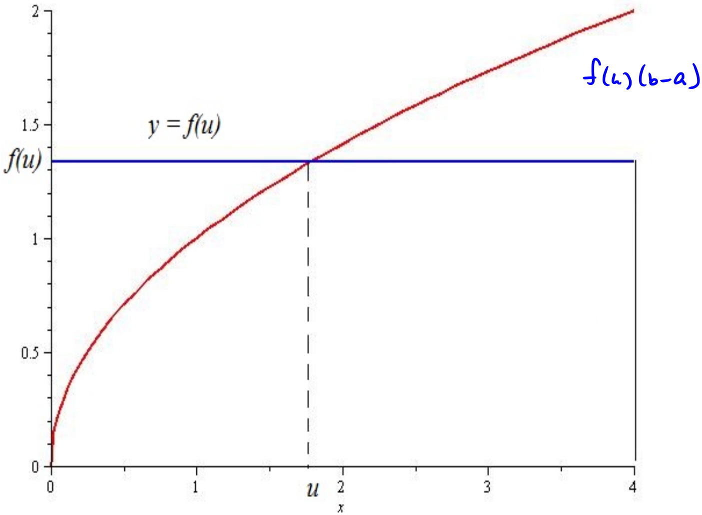


Figure: The area under the red curve is equal to the area in the rectangle with the blue top.

Antiderivatives: e^{ax} and a^x

If a is a nonzero constant

$$\frac{d}{dx}\left(\frac{1}{a}e^{ax}\right) = e^{ax}.$$

Hence $\frac{1}{a}e^{ax}$ is an antiderivative of e^{ax}.

If $a > 0$ and $a \neq 1$, then

$$\frac{d}{dx}\left(\frac{1}{\ln a}a^x\right) = a^x.$$

Hence $\frac{1}{\ln a}a^x$ is an antiderivative of a^x.

Evaluate Each Integral

(a) $\int_1^2 (e^{2t} + 2^t)\, dt$

$$= \frac{1}{2}e^{2t} + \frac{1}{\ln 2}2^t \Big|_1^2$$

$$= \frac{1}{2}e^{2\cdot 2} + \frac{1}{\ln 2}2^2 - \left(\frac{1}{2}e^{2\cdot 1} + \frac{1}{\ln 2}2^1\right)$$

$$= \frac{1}{2}e^4 + \frac{4}{\ln 2} - \frac{1}{2}e - \frac{2}{\ln 2} = \frac{1}{2}e^4 + \frac{2}{\ln 2} - \frac{1}{2}e$$

(b) $\int_1^0 (2x-1)^2\,dx$

$(2x-1)^2 = 4x^2 - 4x + 1$

$$= \int_1^0 (4x^2 - 4x + 1)\,dx$$

$$= 4\frac{x^3}{3} - 4\frac{x^2}{2} + x \Big|_1^0$$

$$= \frac{4}{3}0^3 - 2\cdot 0^2 + 0 - \left(\frac{4}{3}\cdot 1^3 - 2\cdot 1^2 + 1\right)$$

$$0 - \left(\frac{4}{3} - 2 + 1\right) = \frac{-1}{3}$$

Question

(c) $\int_0^2 x^2(4x+6)\,dx$ $= \int_0^2 (4x^3 + 6x^2)\,dx$

$x^4 + 2x^3 \Big|_0^2 = 2^4 + 2\cdot 2^3 - (0^4 + 2\cdot 0^3)$

$= 16 + 16 = 32$

(a) 32

(b) $\frac{160}{3}$

(c) $\frac{68}{3}$

(d) 68

(d) $\int_0^{\pi/4} \tan^2\theta \, d\theta$

Recall $\tan^2\theta + 1 = \sec^2\theta$

$\Rightarrow \tan^2\theta = \sec^2\theta - 1$

$$= \int_0^{\pi/4} (\sec^2\theta - 1)\, d\theta$$

$$= \tan\theta - \theta \Big|_0^{\pi/4}$$

$$= \tan\frac{\pi}{4} - \frac{\pi}{4} - (\tan 0 - 0)$$

$$= 1 - \pi/4 - 0 \qquad = 1 - \pi/4$$

Question

(e) $\displaystyle\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^2 x}$ $= \displaystyle\int_{\pi/4}^{\pi/2} \csc^2 x \, dx = -\cot x \Big|_{\pi/4}^{\pi/2}$

$= -\cot \pi/2 - (-\cot \pi/4)$

(a) -1

(b) 1 $= 0 - (-1) = 1$

(c) 0

(d) This can't be evaluated without more advanced techniques.

(f) $\displaystyle\int_{-6}^{-2} \frac{x^{3/2} + 4x^3 + 2}{x}\,dx$

$$= \int_{-6}^{-2} \left(\frac{x^{3/2}}{x} + \frac{4x^3}{x} + \frac{2}{x} \right) dx$$

$$= \int_{-6}^{-2} \left(x^{1/2} + 4x^2 + \frac{2}{x} \right) dx$$

$$= \frac{x^{3/2}}{3/2} + 4\,\frac{x^3}{3} + 2\ln|x| \Big|_{-6}^{-2}$$

$$= \frac{2}{3}(-2)^{3/2} + \frac{4}{3}(-2)^3 + 2\ln|-2| -$$

$$\left(\frac{2}{3}(-6)^{3/2} + \frac{4}{3}(-6)^3 + 2\ln|-6|\right)$$

Woops! poor planning on my part!

$(-2)^{3/2}$ and $(-6)^{3/2}$ are not real numbers.