November 28 Math 2306 sec. 53 Fall 2018

Section 18: Sine and Cosine Series

Functions with Symmetry: If f is even on $(-p, p)$, then the Fourier series of f has only constant and cosine terms. Moreover

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

where

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx$$

and

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$$

Fourier Series of an Odd Function

If f is odd on $(-p,p)$, then the Fourier series of f has only sine terms. Moreover

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

where

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$$

Half Range Sine and Half Range Cosine Series

Suppose f is only defined for $0 < x < p$. We can **extend** f to the left, to the interval $(-p, 0)$, as either an even function or as an odd function. Then we can express f with **two distinct** series:

Half range cosine series $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$

where $\quad a_0 = \frac{2}{p}\int_0^p f(x)\,dx \quad$ and $\quad a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$

Half range sine series $\quad f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$

where $\quad b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$

Extending a Function to be Odd

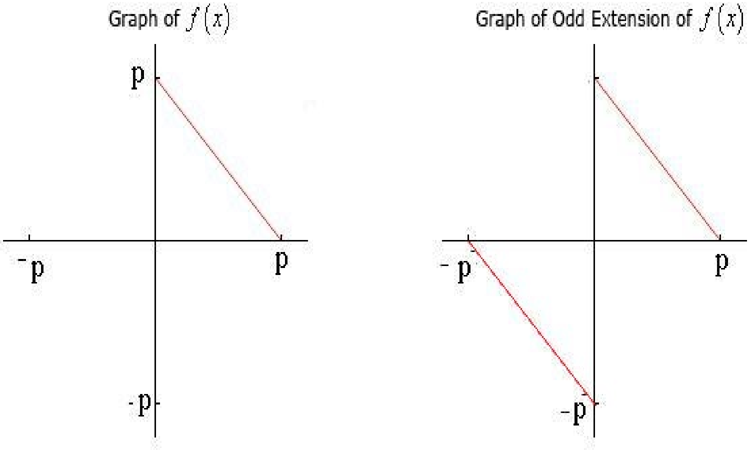

Figure: $f(x) = p - x$, $0 < x < p$ together with its **odd** extension.

Extending a Function to be Even

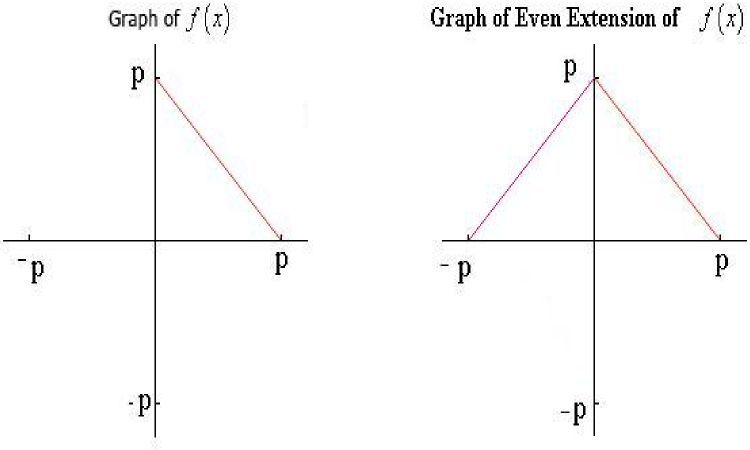

Figure: $f(x) = p - x$, $0 < x < p$ together with its **even** extension.

Find the Half Range Sine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

$p = 2$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$u = 2 - x \qquad du = -dx$

$dv = \sin\left(\frac{n\pi x}{2}\right)dx$

$v = \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$

$$= \frac{-2(2-x)}{n\pi}\cos\left(\frac{n\pi x}{2}\right) - \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{2}\right)\Bigg|_0^2$$

$$= \frac{-2(2\cdot 2)}{n\pi}\cos\left(\frac{n\pi 2}{2}\right) - \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi 2}{2}\right)$$

$$\overset{''}{0} \qquad \overset{''}{0}$$

$$- \left(\frac{-2(2-0)}{n\pi}\cos(0) - \left(\frac{2}{n\pi}\right)^2 \sin(0)\right)$$

$$\overset{''}{0}$$

$$= \frac{-(-2)(2)}{n\pi}\cdot 1$$

$$b_n = \frac{4}{n\pi}$$

The half range sine series is

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$$

Find the Half Range Cosine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 (2-x)\,dx = 2x - \frac{x^2}{2}\Big|_0^2 = 4 - 2 - 0 = 2$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

again integrate by parts

$$= \frac{2(2-x)}{n\pi}\sin\left(\frac{n\pi x}{2}\right) - \left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi x}{2}\right)\Bigg|_0^2$$

$$= \underbrace{\frac{2(2-2)}{n\pi}\sin\left(\frac{n\pi 2}{2}\right)}_{=0} - \left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi 2}{2}\right) -$$

$$\left(\underbrace{\frac{2(2-0)}{n\pi}\sin(0)}_{=0} - \left(\frac{2}{n\pi}\right)^2\cos(0)\right)$$

$$= -\left(\frac{2}{n\pi}\right)^2 \cos(n\pi) + \left(\frac{2}{n\pi}\right)^2$$

$$= \frac{4}{n^2\pi^2}\left(1 - (-1)^n\right)$$

The series is

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)$$

Example Continued...

We have two different half range series:

$$\text{Half range sine:} \quad f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$$

$$\text{Half range cosine:} \quad f(x) = 1 + \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right).$$

We have two different series representations for this function each of which converge to $f(x)$ on the interval $(0, 2)$. The following plots show graphs of f along with partial sums of each of the series. When we plot over the interval $(-2, 2)$ we see the two different symmetries. Plotting over a larger interval such as $(-6, 6)$ we can see the periodic extensions of the two symmetries.

Plots of f with Half range series

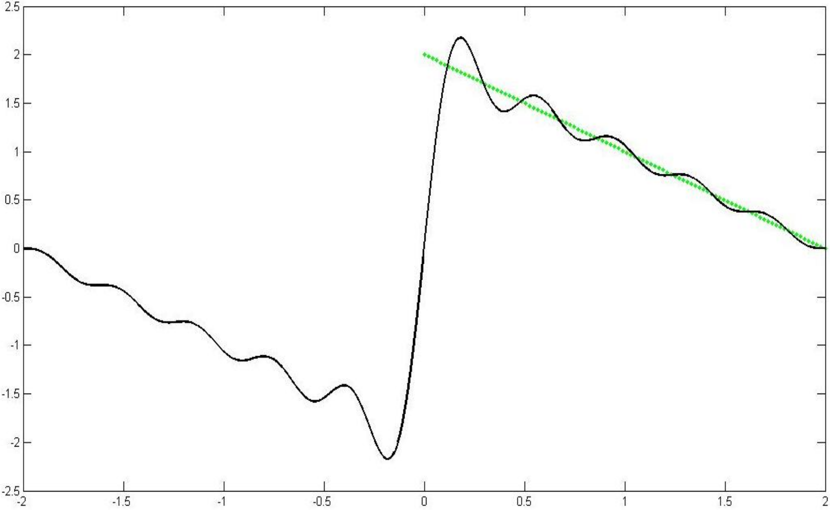

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series.

Plots of f with Half range series

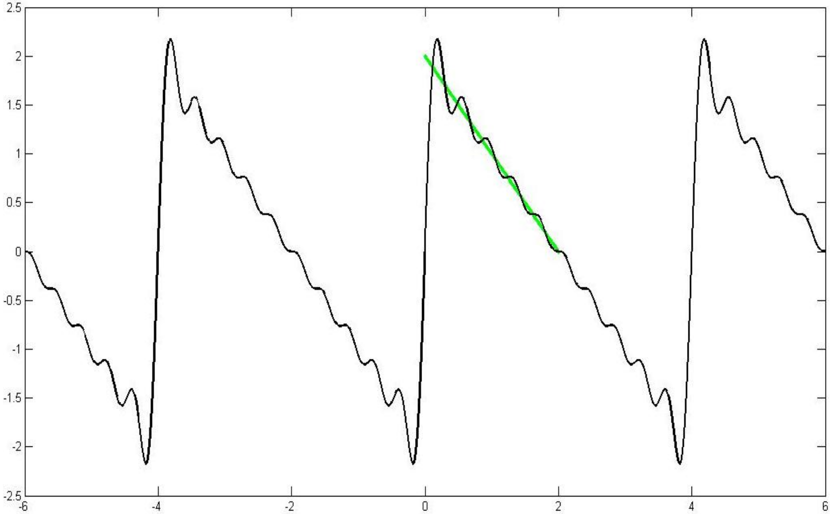

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series, and the series plotted over $(-6, 6)$

Plots of f with Half range series

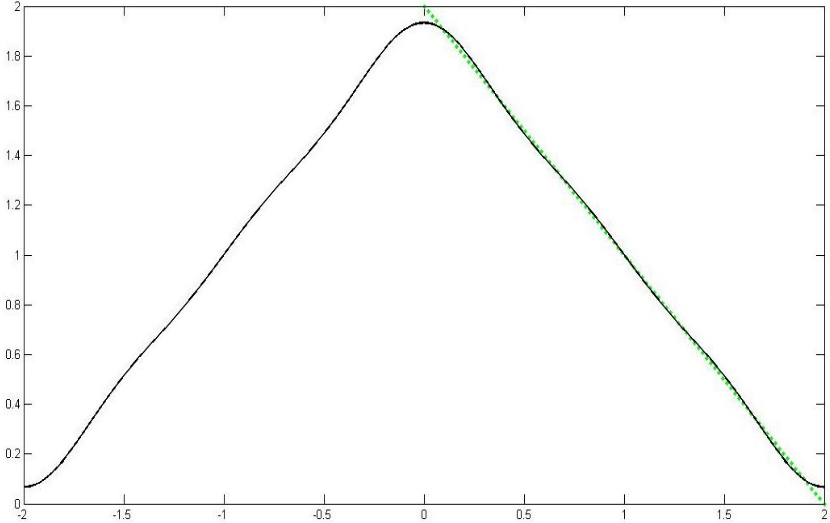

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series.

Plots of f with Half range series

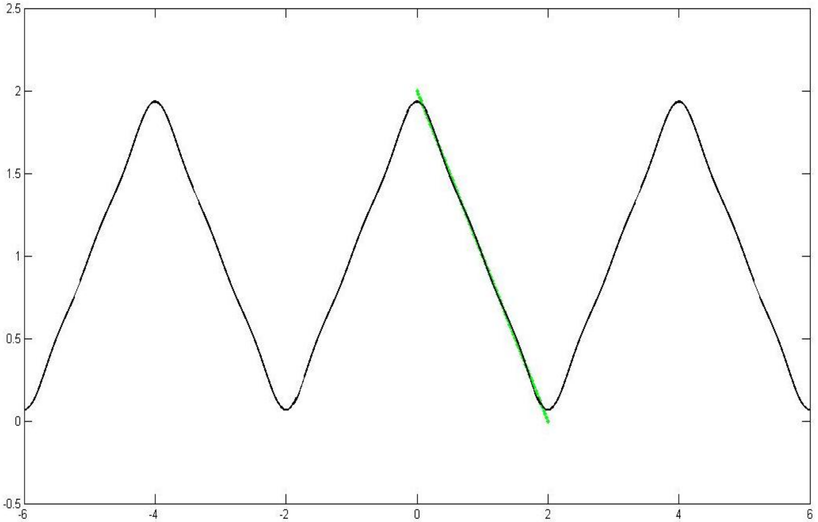

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series, and the series plotted over $(-6, 6)$