Section 5.3: Diagonalization

Determine the eigenvalues of the matrix $D^3$ where $D = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$. 
Diagonal Matrices

Recall: A matrix $D$ is diagonal if it is both upper and lower triangular (its only nonzero entries are on the diagonal).

Note: If $D$ is diagonal with diagonal entries $d_{ii}$, then $D^k$ is diagonal with diagonal entries $d_{ii}^k$ for positive integer $k$. Moreover, the eigenvalues of $D$ are the diagonal entries.
Powers and Similarity

Show that if $A$ and $B$ are similar, with similarity transformation matrix $P$, then $A^k$ and $B^k$ are similar with the same matrix $P$. 
Diagonalizability

**Definition:** An $n \times n$ matrix $A$ is called **diagonalizable** if it is similar to a diagonal matrix $D$. That is, provided there exists a nonsingular matrix $P$ such that $D = P^{-1}AP$—i.e. $A = PDP^{-1}$.

**Theorem:** The $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors. In this case, the matrix $P$ is the matrix whose columns are the $n$ linearly independent eigenvectors of $A$. 
Example

Diagonalize the matrix $A$ if possible. $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$
Example

Diagonalize the matrix $A$ if possible. $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$
Theorem (a second on diagonalizability)

Recall: (sec. 5.1) If $\lambda_1$ and $\lambda_2$ are distinct eigenvalues of a matrix, the corresponding eigenvectors are linearly independent.

Theorem: If the $n \times n$ matrix $A$ has $n$ distinct eigenvalues, then $A$ is diagonalizable.

Note: This is a sufficiency condition, not a necessity condition. We’ve already seen a matrix with a repeated eigenvalue that was diagonalizable.
Theorem (a third on diagonalizability)

**Theorem:** Let $A$ be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.

(a) The geometric multiplicity (dimension of the eigenspace) of $\lambda_k$ is less than or equal to the algebraic multiplicity of $\lambda_k$.

(b) The matrix is diagonalizable if and only if the sum of the geometric multiplicities is $n$—i.e. the sum of dimensions of all eigenspaces is $n$ so that there are $n$ linearly independent eigenvectors.

(c) If $A$ is diagonalizable, and $B_k$ is a basis for the eigenspace for $\lambda_k$, then the collection (union) of bases $B_1, \ldots, B_p$ is a basis for $\mathbb{R}^n$.

**Remark:** The union of the bases referred to in part (c) is called an **eigenvector basis** for $\mathbb{R}^n$. (Of course, one would need to reference the specific matrix.)
Example

Diagonalize the matrix if possible. \( A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \).
Example Continued...

Find $A^4$ where $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$. 
Section 5.4: Eigenvectors and Linear Transformations

Recall: A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ can always be written in terms of a matrix product $T(x) = Ax$ where $A$ is the standard matrix for $T$.

Questions: If $A$ happens to be diagonalizable ($A$ similar to $D$) is there a way to rewrite $T$ in terms of $D$ to take advantage of the nice diagonal matrix?

If we replace $\mathbb{R}^n$ and $\mathbb{R}^m$ with other (finite dimensional) vectors spaces (e.g. $\mathbb{P}_n$ or $M^{2\times2}$) can we still write $T$ in terms of a matrix?
The second question

Recall: If $V$ is an $n$ dimensional vector space with ordered basis $\mathcal{B} = \{b_1, \ldots, b_n\}$, then for each $x$ in $V$ we can consider its coordinate vector, which is a vector in $\mathbb{R}^n$, $[x]_\mathcal{B}$.

Schematic: Let $T : V \rightarrow W$ be linear with $V$ and $W$ vector spaces of dimension $n$ and $m$, respectively. Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis for $V$, and $\mathcal{C} = \{c_1, \ldots, c_m\}$ be a basis for $W$. Then build a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ that captures the relationship

$$[x]_\mathcal{B} \mapsto [T(x)]_\mathcal{C}.$$ 

This mapping has a matrix $M$. 
Matrix of a Linear Transformation

Let \( T : V \rightarrow W \). For \( \mathbf{x} \) in \( V \) suppose

\[
[\mathbf{x}]_B = (r_1, r_2, \ldots, r_n)
\]

so that

\[
T(\mathbf{x}) = r_1 T(\mathbf{b}_1) + r_2 T(\mathbf{b}_2) + \cdots + r_n T(\mathbf{b}_n).
\]

Next, using the coordinate mapping on \( W \)

\[
[T(\mathbf{x})]_C = r_1[T(\mathbf{b}_1)]_C + r_2[T(\mathbf{b}_2)]_C + \cdots + r_n[T(\mathbf{b}_n)]_C.
\]

This is a vector equation in \( \mathbb{R}^m \)! So we can write it as a matrix product

\[
[T(\mathbf{x})]_C = M[\mathbf{x}]_B
\]

where the columns of \( M \) are the vectors \([T(\mathbf{b}_i)]_C\) for \( i = 1, \ldots, n \). \( M \) is called the **matrix for \( T \) relative to the bases** \( B \) and \( C \).
Matrix of a Linear Transformation

\[ x \in V \]

\[ x \rightarrow [x]_B \]

\[ [x]_B \in \mathbb{R}^n \]

\[ T \]

\[ T(x) \]

\[ [T(x)]_C = M[x]_B \]

\[ [T(x)]_C \in \mathbb{R}^m \]
Example

Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2, c_3\}$ be bases for $V$ and $W$ respectively. And suppose the linear transformation $T : V \rightarrow W$ is defined by the properties

$$T(b_1) = c_1 - 3c_2 + 4c_3,$$
and $$T(b_2) = 5c_2 - 2c_3.$$

Find the matrix $M$ for $T$ relative to the bases $B$ and $C$. 

An Example with $V = W$

Find the matrix for $T : \mathbb{P}_2 \rightarrow \mathbb{P}_2$ relative to the basis $B = \{1, t, t^2\}$ where $T$ is defined by

$$T(p_0 + p_1 t + p_2 t^2) = p_1 - 3p_1 t + (p_0 - p_2)t^2.$$ 

Use the results to find $T(2t^2 + t - 5)$. 
Transformation from $V$ to $V$

**Definition:** If $V$ is an $n$ dimensional vector space and $T : V \longrightarrow V$ is linear, and if a single basis $\mathcal{B}$ is used to construct a matrix for $T$, then this matrix is called **the matrix for $T$ relative to $\mathcal{B}$** or simply **the $\mathcal{B}$ matrix of $T$**. It will be denoted by

$$[T]_\mathcal{B}.$$
Example $V = \mathbb{R}^2$

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$T(x_1, x_2) = (3x_1 - 2x_2, x_1 + 4x_2).$$

Find the $B$ matrix of $T$ for the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right\}.$$

\(^1\)Remember that $[x] = P_B^{-1}x$ where $P_B = [b_1 \ b_2]$. 
Example Continued...

Find the standard matrix $A$ for $T(x_1, x_2) = (3x_1 - 2x_2, x_1 + 4x_2)$, and verify that

$$[T]_B = P_B^{-1} A P_B.$$
Theorem

**Theorem:** If $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear transformation with standard matrix $A$, and $B$ is an ordered basis for $\mathbb{R}^n$, then the $B$ matrix of $T$ is

$$ [T]_B = P_B^{-1} A P_B. $$

**Corollary:** If $A$ is diagonalizable with $D = P^{-1} A P$, and $B$ is the basis of $\mathbb{R}^n$ consisting of the columns of $P$, then the $B$ matrix of $T$ is the diagonal matrix

$$ [T]_B = D. $$
Example

Find a basis $B$ for $\mathbb{R}^2$ such that the $B$ matrix of $T$ is diagonal where

$$T(\mathbf{x}) = A\mathbf{x}, \quad \text{and} \quad A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}.$$