November 28 Math 3260 sec. 58 Fall 2017

Section 5.3: Diagonalization

Determine the eigenvalues of the matrix D^3 where $D = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$.

$$D^2 = DD = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 16 \end{bmatrix}$$

$$D^3 = D^2 D = \begin{bmatrix} 9 & 0 \\ 0 & 16 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 27 & 0 \\ 0 & -64 \end{bmatrix}$$

The eigenvalues of D^3 are $\lambda_1 = 27$ and $\lambda_2 = -64$

Diagonal Matrices

Recall: A matrix D is diagonal if it is both upper and lower triangular (its only nonzero entries are on the diagonal).

Note: If D is diagonal with diagonal entries d_{ii}, then D^k is diagonal with diagonal entries d_{ii}^k for positive integer k. Moreover, the eigenvalues of D are the diagonal entries.

Powers and Similarity

Show that if A and B are similar, with similarity tranformation matrix P, then A^k and B^k are similar with the same matrix P.

B is similar to A if $B = P^{-1}AP$ for some nonsingular matrix P. Let's see that B^2 is similar to A^2.

$$B^2 = BB = (P^{-1}AP)(P^{-1}AP)$$

$$= P^{-1}A(PP^{-1})AP = P^{-1}AIAP$$

$$= P^{-1}AAP = P^{-1}A^2P$$

Suppose $B^k = P^{-1}A^kP$ for some $k \geq 1$.

Then

$$B^{k+1} = B^kB = (P^{-1}A^kP)(P^{-1}AP)$$

$$= P^{-1}A^kAP$$

$$= P^{-1}A^{k+1}P$$

Diagonalizability

Defintion: An $n \times n$ matrix A is called **diagonalizable** if it is similar to a diagonal matrix D. That is, provided there exists a nonsingular matrix P such that $D = P^{-1}AP$—i.e. $A = PDP^{-1}$.

Theorem: The $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In this case, the matrix P is the matrix whose columns are the n linearly independent eigenvectors of A.

Example

Diagonalize the matrix A if possible. $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} - 3\begin{vmatrix} -3 & -3 \\ 3 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} -3 & -5-\lambda \\ 3 & 3 \end{vmatrix}$$

$$= (1-\lambda)\left[(-5-\lambda)(1-\lambda)+9\right] - 3\left(-3(1-\lambda)+9\right) + 3\left(-9-3(-5-\lambda)\right)$$

$$= (1-\lambda)\left[\lambda^2+4\lambda-5+9\right] - 3\left(3\lambda+6\right) + 3\left(3\lambda+6\right)$$

$$= (1-\lambda)\left(\lambda^2+4\lambda+4\right) = (1-\lambda)(\lambda+2)^2$$

$\det(A-\lambda I)=0$ if $\lambda=1$ or $\lambda=-2$

$$A-1I = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} x_1 = x_3 \\ x_2 = -x_3 \\ x_3 - \text{free} \end{array}$$

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$A - (-2)I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2 - x_3$

x_2, x_3 free

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We have 3 linearly independent eigenvectors

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

A matrix P is

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Example

Diagonalize the matrix A if possible. $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)\Big((-6-\lambda)(1-\lambda)+9\Big) - 4\Big(-4(1-\lambda)+9\Big)$$

$$+3\Big(-12-3(-6-\lambda)\Big)$$

$$= (2-\lambda)\left(\lambda^2+5\lambda+3\right) - 4\left(4\lambda+5\right) + 3\left(3\lambda+6\right)$$

$= 2\lambda^2 + 10\lambda + 6 - \lambda^3 - 5\lambda^2 - 3\lambda - 16\lambda - 20 + 9\lambda + 18$

$= -\lambda^3 - 3\lambda^2 + 4$

$-1 - 3 + 4 = 0 \Rightarrow$ 1 is a root

$= (1-\lambda)(\lambda^2 + 4\lambda + 4) = (1-\lambda)(\lambda+2)^2$

The eigen values are $\lambda_1 = 1$ and $\lambda_2 = -2$.

$$A - 1I = \begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_3$

$x_2 = -x_3$

x_3 - free

$$\vec{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$A - (-2)I = \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2$

x_2 - free

$x_3 = 0$

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

There are only two linearly independent eigenvectors.

A is not diagonalizable.

Suppose we set up $A-\lambda I$

and

$$A-\lambda I \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{x}=\vec{0}$$

Theorem (a second on diagonalizability)

Recall: (sec. 5.1) If λ_1 and λ_2 are distinct eigenvalues of a matrix, the corresponding eigenvectors are linearly independent.

Theorem: If the $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Note: This is a *sufficiency* condition, not a *necessity* condition. We've already seen a matrix with a repeated eigenvalue that was diagonalizable.

Theorem (a third on diagonalizability)

Theorem: Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.

(a) The geometric multiplicity (dimension of the eigenspace) of λ_k is less than or equal to the algebraic multiplicity of λ_k.

(b) The matrix is diagonalizable if and only if the sum of the geometric multiplicities is n—i.e. the sum of dimensions of all eigenspaces is n so that there are n linearly independent eigenvectors.

(c) If A is diagonalizable, and $\mathcal{B}_k$ is a basis for the eigenspace for λ_k, then the collection (union) of bases $\mathcal{B}_1, \ldots, \mathcal{B}_p$ is a basis for $\mathbb{R}^n$.

Remark: The union of the bases referred to in part (c) is called an **eigenvector basis** for $\mathbb{R}^n$. (Of course, one would need to reference the specific matrix.)

Example

Diagonalize the matrix if possible. $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 12$$

$$= \lambda^2 - 3\lambda + 2 - 12$$

$$= \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$$

$\lambda_1 = 5$, $\lambda_2 = -2$

Find eigenvectors

$$A - 5I = \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -\frac{3}{4} \\ 0 & 0 \end{bmatrix}$$

$x_1 = \frac{3}{4} x_2$

x_2 - free

$$\vec{x} = x_2 \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} = \frac{x_2}{4} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$A - (-2)I = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$x_1 = -x_2$

x_2 - free

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

For $P = \begin{bmatrix} 3 & -1 \\ 4 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$

Example Continued...

Find A^4 where $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$.

We know that $D = P^{-1}AP$

where $D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$, $P = \begin{bmatrix} 3 & -1 \\ 4 & 1 \end{bmatrix}$

$$D^4 = \begin{bmatrix} 625 & 0 \\ 0 & 16 \end{bmatrix}$$

$$D^4 = P^{-1} A^4 P$$

$$\Rightarrow \quad A^4 = P D^4 P^{-1}$$

$$= \begin{bmatrix} 277 & 261 \\ 348 & 364 \end{bmatrix}$$

Suppose A is an $n \times n$ matrix,

can we define e^A.

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \cdots$$

This is just an example of where computing lots of powers of a matrix may have application.