November 2 MATH 1113 sec. 51 Fall 2018

Section 6.4: Radian Measure

Degree measure is sometimes used in technical fields (surveying and engineering). But degrees complicate many mathematical computations. We prefer another measure that is in some sense *unitless* [1].

Radians: (Rad) An angle is measured in radians in relation to a unit circle (circle of radius 1).

An angle $\theta = 1$ radian if the angle subtends an arc in a unit circle of length 1.

[1] We'll still call them units, but it will become more clear that they aren't units in the traditional sense.

A Radian

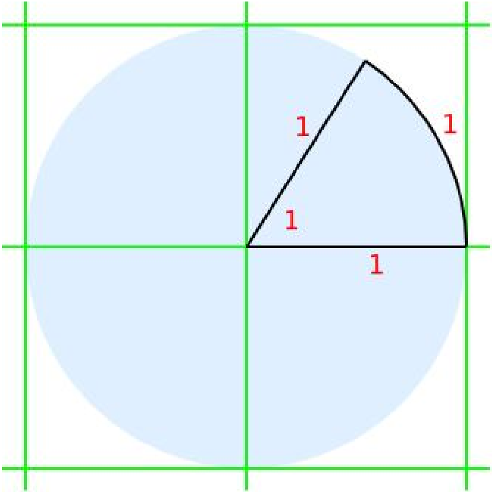


Figure: One Radian: The length of the arc equals the radius of the circle.

Radian Measure

The arc-length of a whole unit circle is 2π. So...

There are 2π radians in one circle (a little more than 6 of them)!

Converting Between Degrees & Radians

Since $360^\circ = 2\pi$ rad, we get the following conversion factors:

$$1^\circ = \frac{\pi}{180} \text{ rad} \qquad \text{and} \qquad 1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ$$

Remark: If an angle doesn't have the degree symbol $^\circ$ next to it, it is assumed to be in radians!

Converting Between Angle Measures

- To convert from degrees to radians, multiply by

$$\frac{\pi}{180}.$$

- To convert from radians to degrees, multiply by

$$\frac{180}{\pi} \quad \text{and insert the symbol} \quad \circ \, .$$

Example

Convert each angle measure to the other *units*.

(a) 45°

$$45^\circ \cdot \frac{\pi}{180^\circ} = \frac{45}{180}\pi = \frac{\pi}{4}$$

(b) $-\frac{\pi}{6}$

$$\left(-\frac{\pi}{6} \cdot \frac{180}{\pi}\right)^\circ = -\frac{180}{6}^\circ = -30^\circ$$

(b) 30

$$\left(30 \cdot \frac{180}{\pi}\right)^\circ = \frac{5400}{\pi}^\circ$$

Question

If $\theta = -210°$, then in radians

(a) $\theta = \frac{7\pi}{6}$

(b) $\theta = -\frac{7\pi}{6}$

$-210 \cdot \frac{\pi}{180}$

(c) $\theta = \frac{6\pi}{7}$

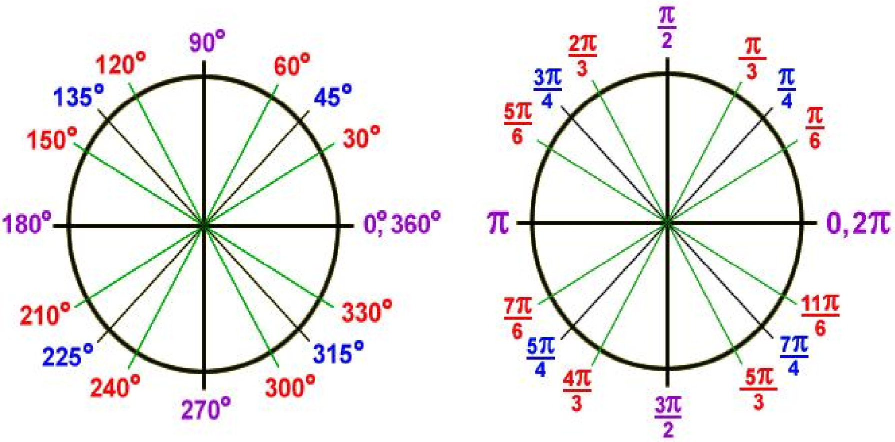
(d) $\theta = -\frac{6\pi}{7}$

(e) there's no such thing as a negative angle

Some Common Angles: Degree and Radian

θ°	θ rad
0°	0
30°	$\frac{\pi}{6}$
45°	$\frac{\pi}{4}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
180°	π
270°	$\frac{3\pi}{2}$
360°	2π

Angles With *Nice* Reference Angles



We Recall A Few Terms

Some special names for angles include:

- An **acute** angle is between 0° and 90° (0 and $\frac{\pi}{2}$).
- An **obtuse** angle is between 90° and 180° ($\frac{\pi}{2}$ and π).
- A **right** angle has measure 90° ($\frac{\pi}{2}$).
- A **reflex** angle has measure between 180° (π) and 360° (2π).
- **Quadrantal** angles are integer multiples of 90° ($\frac{\pi}{2}$)
- A **straight** angle has measure 180° (π).

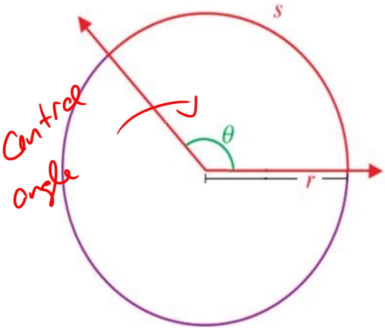
(Of course, not all angles fit into one of these categories.)

Arclength Formula

Given a circle of radius r, the length s of the arc subtended by the (positive) central angle θ (**in radians**) is given by

$$s = r\theta.$$

The area of the resulting sector is $A_{sector} = \frac{1}{2}r^2\theta$.



$A_{circle} = \pi r^2$

Sector fraction of the circle is $\frac{\theta}{2\pi}$

Area sector = fraction of Area circle

$A_{sector} = \pi r^2\left(\frac{\theta}{2\pi}\right) = \frac{1}{2}r^2\theta$

Example

A circle of radius 12 meters has a sector given by a central angle of 135°. Find the associated arc length and the area of the sector.

Arc length $s = r\theta$ and Area $A_{sector} = \frac{1}{2}r^2\theta$

These formulas require θ in radians

$\theta = 135^0 \implies \theta = 135 \cdot \frac{\pi}{180}$ radians

$= \frac{3\pi}{4}$

So

$s = (12\ m)\left(\frac{3\pi}{4}\right) = 9\pi\ m$

$$A_{sector} = \frac{1}{2}(12m)^2\left(\frac{3\pi}{4}\right) = \frac{144\cdot 3\pi}{8}\ m^2$$

$$= 18\cdot 3\pi\ m^2$$

$$= 54\pi\ m^2$$

Question

An industrial clock has a face that is 3 ft in **diameter.** What is the area of the sector between the 12 and the 4 hour markings?

(a) $\dfrac{9\pi}{2}$ ft^2

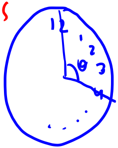
(b) $\dfrac{3\pi}{2}$ ft^2

(c) $\dfrac{3\pi}{4}$ ft^2

(d) 3π ft^2

(e) can't be determined without more information

$\frac{4 \text{ out of}}{12 \text{ hours}} = \frac{1}{3}$



$\theta = \frac{2\pi}{3}$ ($\frac{1}{3}$ of the circle)

$r = \frac{1}{2} d = \frac{3}{2}$ ft

$A_{sector} = \frac{1}{2} r^2 \theta = \frac{1}{2}\left(\frac{3}{2}\text{ ft}\right)^2 \cdot \frac{2\pi}{3}$

$= \frac{9}{4}\text{ ft}^2 \cdot \frac{\pi}{3} = \frac{3\pi}{4}\text{ ft}^2$

Motion on a Circle: Angular & Linear Speed

Definition: (angular speed) If an object moves along the arc of a circle through a central angle θ in the time t, the angular speed is denoted by ω (lower case omega) and is defined by

$$\omega = \frac{\theta}{t} = \frac{\text{angle moved through}}{\text{time}}.$$

θ in radians only

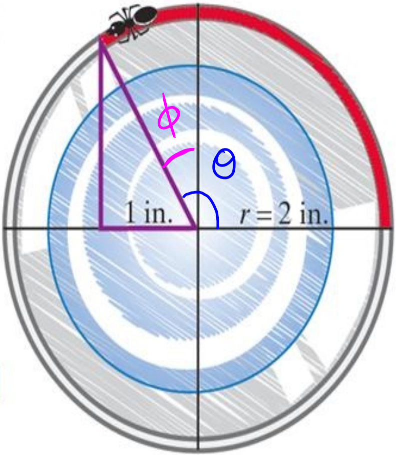
Definition: (linear speed) If the circle has radius r, then the distance traveled is the arclength $s = r\theta$. The linear speed is denoted by ν (lower case nu) and is defined by

$$\nu = \frac{s}{t} = \frac{r\theta}{t} = r\omega.$$

Note that this is distance (s) per unit time (t).

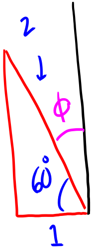
Example

Suppose an ant crawls along the rim of a circular glass with radius 2 inches, and traverses the arc indicated in red in 20 seconds.
What are the angular and linear speeds of the ant, and how far does it travel?



We need to find the central angle θ.

$$\theta = \frac{\pi}{2} + \phi$$



The red triangle is a 30°-60°-90° right triangle. ϕ is the complement of 60°, so $\phi = 30^\circ$.

$$\phi = \frac{\pi}{6}$$

Hence $\theta = \frac{\pi}{2} + \frac{\pi}{6} = \frac{3\pi}{6} + \frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$

Angular speed

$$\omega = \frac{\theta}{t} = \frac{\frac{2\pi}{3}}{20\,\text{sec}} = \frac{2\pi}{3\cdot 20}\,\frac{1}{\text{sec}} = \frac{\pi}{30}\,\frac{1}{\text{sec}}$$

Linear Speed

$$v = \frac{s}{t} = r\omega = 2\text{in}\left(\frac{\pi}{30}\ \frac{1}{\text{sec}}\right) = \frac{\pi}{15}\ \frac{\text{in}}{\text{sec}}$$

The ant travels

$$s = r\theta = 2\text{in}\left(\frac{2\pi}{3}\right) = \frac{4\pi}{3}\ \text{in}$$