November 2 MATH 1113 sec. 52 Fall 2018

Section 6.4: Radian Measure

Degree measure is sometimes used in technical fields (surveying and engineering). But degrees complicate many mathematical computations. We prefer another measure that is in some sense *unitless* [1].

Radians: (Rad) An angle is measured in radians in relation to a unit circle (circle of radius 1).

An angle $\theta = 1$ radian if the angle subtends an arc in a unit circle of length 1.

[1] We'll still call them units, but it will become more clear that they aren't units in the traditional sense.

A Radian

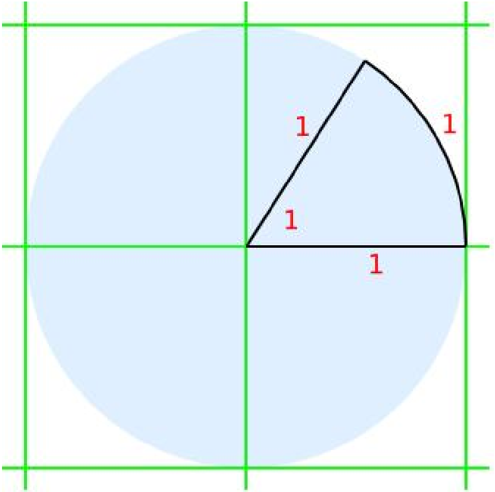


Figure: One Radian: The length of the arc equals the radius of the circle.

Radian Measure

The arc-length of a whole unit circle is 2π. So...

There are 2π radians in one circle (a little more than 6 of them)!

Converting Between Degrees & Radians

Since $360^\circ = 2\pi$ rad, we get the following conversion factors:

$$1^\circ = \frac{\pi}{180}\text{ rad} \qquad \text{and} \qquad 1\text{ rad} = \left(\frac{180}{\pi}\right)^\circ$$

Remark: If an angle doesn't have the degree symbol $^\circ$ next to it, it is assumed to be in radians!

Converting Between Angle Measures

- To convert from degrees to radians, multiply by
$$\frac{\pi}{180}.$$

- To convert from radians to degrees, multiply by
$$\frac{180}{\pi} \quad \text{and insert the symbol} \quad \circ .$$

Example

Convert each angle measure to the other *units*.

(a) 45°

$$45^\circ \frac{\pi}{180^\circ} = \frac{45\pi}{180} = \frac{\pi}{4}$$

(b) $-\frac{\pi}{6}$

$$\left(-\frac{\pi}{6} \cdot \frac{180}{\pi}\right)^\circ = \left(\frac{-180}{6}\right)^\circ = -30^\circ$$

(b) 30

$$\left(30 \cdot \frac{180}{\pi}\right)^\circ = \frac{5400}{\pi}^\circ$$

Question

If $\theta = -210^\circ$, then in radians

(a) $\theta = \frac{7\pi}{6}$

(b) $\theta = -\frac{7\pi}{6}$

(c) $\theta = \frac{6\pi}{7}$

(d) $\theta = -\frac{6\pi}{7}$

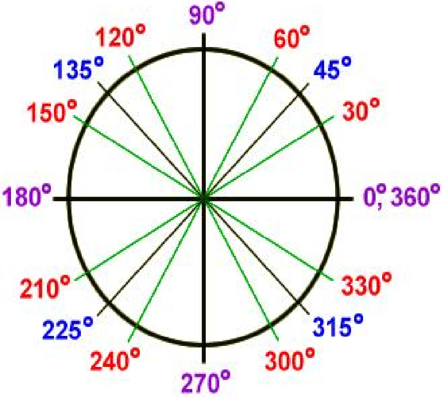
(e) there's no such thing as a negative angle

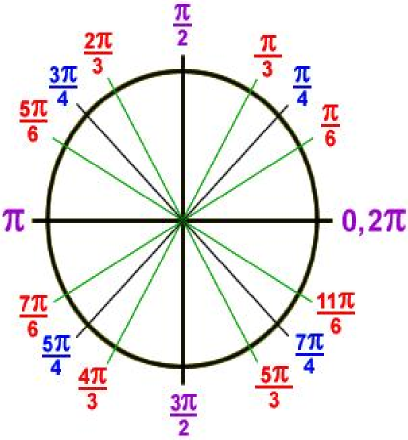
$\frac{-210}{180}\pi$

Some Common Angles: Degree and Radian

θ°	θ rad
0°	0
30°	$\frac{\pi}{6}$
45°	$\frac{\pi}{4}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
180°	π
270°	$\frac{3\pi}{2}$
360°	2π

Angles With *Nice* Reference Angles





We Recall A Few Terms

Some special names for angles include:

- An **acute** angle is between 0° and 90° (0 and $\frac{\pi}{2}$).
- An **obtuse** angle is between 90° and 180° ($\frac{\pi}{2}$ and π).
- A **right** angle has measure 90° ($\frac{\pi}{2}$).
- A **reflex** angle has measure between 180° (π) and 360° (2π).
- **Quadrantal** angles are integer multiples of 90° ($\frac{\pi}{2}$)
- A **straight** angle has measure 180° (π).

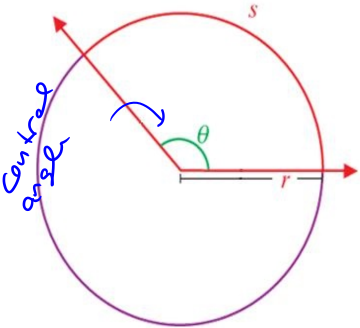
(Of course, not all angles fit into one of these categories.)

Arclength Formula

Given a circle of radius r, the length s of the arc subtended by the (positive) central angle θ (**in radians**) is given by

$$s = r\theta.$$

The area of the resulting sector is $A_{sector} = \frac{1}{2}r^2\theta$.



Area of circle $= \pi r^2$

The sector is a fraction of the circle. The fraction is $\frac{\theta}{2\pi}$ ths

Area of sector = Area circle × fraction

$$A_{sector} = \pi r^2\left(\frac{\theta}{2\pi}\right) = \frac{1}{2}r^2\theta$$

Example

A circle of radius 12 meters has a sector given by a central angle of 135°. Find the associated arc length and the area of the sector.

Arclength $s = r\theta$ and Area Sector $A_{sector} = \frac{1}{2}r^2\theta$

for θ in radians.

Convert θ to radians

$$\theta = 135^\circ \cdot \frac{\pi}{180^\circ} = \frac{3\pi}{4}$$

So the arclength

$$s = (12m)\left(\frac{3\pi}{4}\right) = 9\pi \ m$$

The sector area

$$A_{sector} = \frac{1}{2}(12\,m)^2\left(\frac{3\pi}{4}\right) = 54\pi \ m^2$$

Question

An industrial clock has a face that is 3 ft in **diameter.** What is the area of the sector between the 12 and the 4 hour markings?

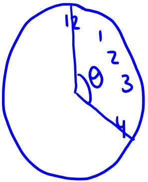
(a) $\frac{9\pi}{2}$ ft^2

(b) $\frac{3\pi}{2}$ ft^2

(c) $\frac{3\pi}{4}$ ft^2

(d) 3π ft^2

(e) can't be determined without more information



$\theta = \frac{2\pi}{3}$

4 out of 12 hours so $\frac{1}{3}$ clock

$\theta = 2\pi \cdot \frac{1}{3} = \frac{2\pi}{3}$

$r = \frac{3}{2}$ ft

$\text{Area} = \frac{1}{2}\left(\frac{3}{2}\text{ft}\right)^2 \cdot \left(\frac{2\pi}{3}\right) = \frac{1}{2} \cdot \frac{9}{4} \cdot \frac{2}{3}\pi \text{ ft}^2$

$= \frac{3\pi}{4} \text{ ft}^2$

Motion on a Circle: Angular & Linear Speed

Definition: (angular speed) If an object moves along the arc of a circle through a central angle θ in the time t, the angular speed is denoted by ω (lower case omega) and is defined by

$$\omega = \frac{\theta}{t} = \frac{\text{angle moved through}}{\text{time}}.$$

Definition: (linear speed) If the circle has radius r, then the distance traveled is the arclength $s = r\theta$. The linear speed is denoted by ν (lower case nu) and is defined by

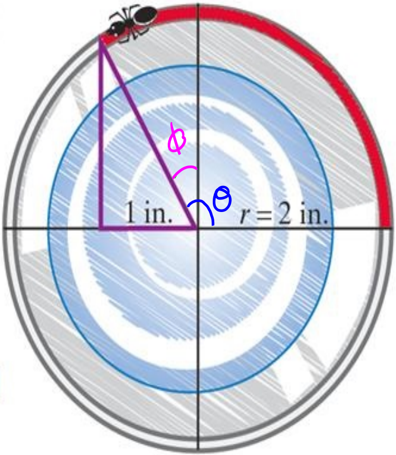
$$\nu = \frac{s}{t} = \frac{r\theta}{t} = r\omega.$$

Note that this is distance (s) per unit time (t).

Example

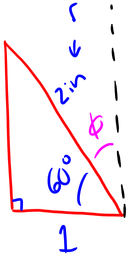
Suppose an ant crawls along the rim of a circular glass with radius 2 inches, and traverses the arc indicated in red in 20 seconds.

What are the angular and linear speeds of the ant, and how far does it travel?



Angular speed $\omega = \frac{\theta}{t}$, linear speed $v = \frac{s}{t} = \frac{r\theta}{t} = r\omega$

$\theta = \frac{\pi}{2} + \phi$



ϕ is the complement of $60°$

The triangle is $30°$-$60°$-$90°$

$$\phi = 30\left(\frac{\pi}{180}\right) = \frac{\pi}{6}$$

$$\theta = \frac{\pi}{2} + \frac{\pi}{6} = \frac{3\pi}{6} + \frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$$

The angular velocity

$$\omega = \frac{\theta}{t} = \frac{\frac{2\pi}{3}}{20\text{ sec}} = \frac{\pi}{30}\ \frac{1}{\text{sec}}$$

The linear velocity

$$v = r\omega = (2\text{in})\left(\frac{\pi}{30}\ \frac{1}{\text{sec}}\right) = \frac{\pi}{15}\ \frac{\text{in}}{\text{sec}}$$

The ant travels

$$s = r\theta = 2\text{in}\left(\frac{2\pi}{3}\right) = \frac{4\pi}{3}\text{ in}$$