Nov. 2 Math 1190 sec. 51 Fall 2016

Section 4.7: Optimization

Optimization problems arise in every field of study and every industry.

- minimize cost and maximize revenue,
- maximize crop yield,
- minimize driving time,
- maximize volume,
- minimize energy

Comes down to finding extrema of some function called the objective

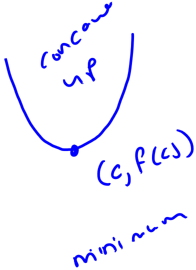
Often, some constraint (extra condition) must simultaneously be satisfied.

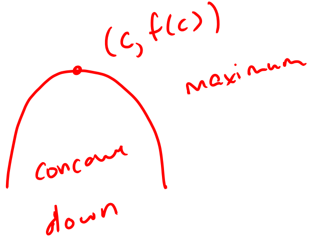
2^{nd} Derivative Test

Recall: If c is a critical number of the function f, and

- $f''(c) > 0$, then f has a local minumum at c

- $f''(c) < 0$, then f has a local maximum at c.





Applied Optimization Example

A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of its sides. What dimensions of the outer rectangle will require the smallest total length of fencing and how much fencing will be needed?

Minimize length of the fence

We're constrained by the given area of 216 m^2

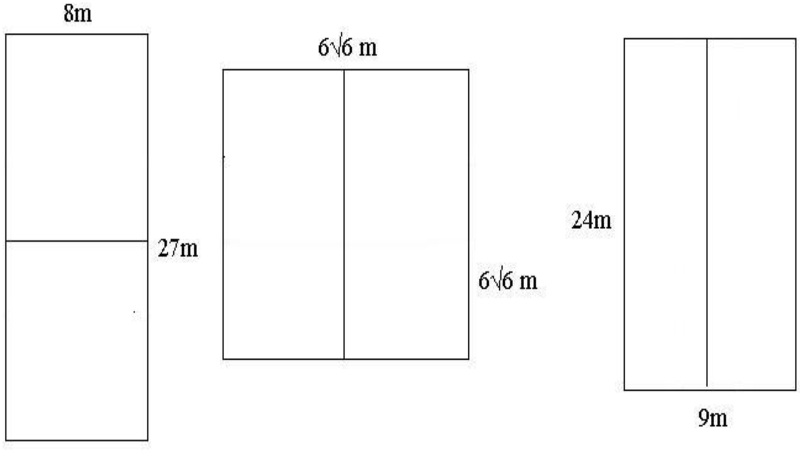
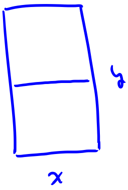


Figure: Different pea patch configuration that all enclose $216m^2$.

A representative pea patch is



Let x and y be the width and length in meters.

Area $A = xy$ and total perimeter $P = 3x + 2y$

Our goal is to minimize P given $A = 216 \text{ m}^2$.

We need P as a function of only one variable.

We can use $A = xy = 216$ to get y in terms of x.

$$y = \frac{216}{x}$$

So $P = 3x + 2\left(\frac{216}{x}\right) = 3x + \frac{432}{x} = 3x + 432 \cdot x^{-1}$

Let's find the critical numbers.

$$P'(x) = 3 - 432x^{-2} = 3 - \frac{432}{x^2}$$

$P'(x)$ is undefined if $x=0$ but as a length we need $x>0$.

$P'(x) = 0 \Rightarrow 3 - \frac{432}{x^2} = 0 \Rightarrow 3 = \frac{432}{x^2} \Rightarrow x^2 = \frac{432}{3}$

$$\Rightarrow x = \pm\sqrt{144} = \pm 12$$

Only the positive root $x=12$ makes sense as a length.

So 12 is a critical number of P, lets verify that this actually minimizes P.

Let's use the 2nd derivative test. $P'(x) = 3 - 432x^{-2}$

$$P''(x) = 2(432)x^{-3} = \frac{2(432)}{x^3}$$

so $P''(12) = \dfrac{2(432)}{(12)^3}$ which is positive.

Thus $x = 12$ minimizes P by the 2nd der. test.

The optimal x is 12m. $y = \frac{216}{x}$, so the optimal

$$y = \frac{216 m^2}{12 m} = 18 m$$. This will require

$$3(12m) + 2(18m) = 36m + 36m = 72m$$

The outer dimensions should be

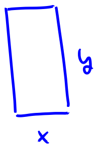
$12m \times 18m$

requiring 72 m of fencing.

Applied Optimization Example

Show that among all rectangles with perimeter 8m, the one with the largest area is a square.

Let's start with a representative rectangle



with length and width x and y meters, respectively. The area A and perimeter P are

$$A = xy \quad \text{and} \quad P = 2x + 2y$$

We need to maximize A given the constraint $P = 8\text{m}$.

* The optimum will be square if we find $x = y$.

From $P = 8m$, $2x + 2y = 8 \Rightarrow 2x = 8 - 2y \Rightarrow x = 4 - y$

The area becomes $A = xy = (4-y)y = 4y - y^2$

Let's find the critical number(s).

$A'(y) = 4 - 2y$ $A'(y)$ is always defined.

$A'(y) = 0 \Rightarrow 4 - 2y = 0 \Rightarrow 2y = 4 \Rightarrow y = 2$.

Let's use the 2nd der. test to see if this maximizes A.

$A''(y) = -2$ so $A''(2) = -2$ which is negative.

Hence $y=2$ maximizes A by the 2^{nd}. der. test.

The optimal $y = 2m$. The optimal $(x = 4-y)$

$$x = 4m - 2m = 2m$$

So the maximum area is when $x = y = 2m$.

The rectangle is a square.

Applied Optimization Example

Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius 10.

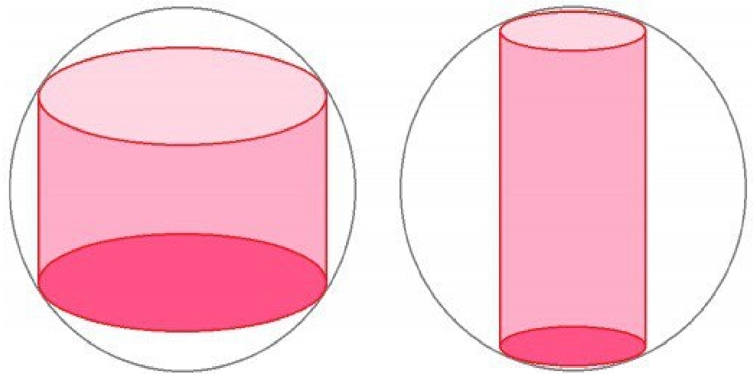
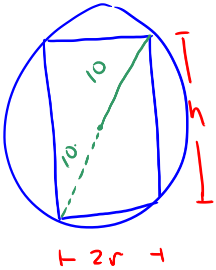


Figure: Different cylinders can be inscribed in the same sphere. We want to find the largest possible one.

If we take a cross section perpendicula to the base of the cylinder.



The volume of a cylinder of radius r and height h is

$$V = \pi r^2 h$$

Since the cylinder is inscribed in the sphere of radius 10

$$h^2 + (2r)^2 = (2 \cdot 10)^2$$

$$\Rightarrow \quad h^2 + 4r^2 = 400$$

We need to maximize $V = \pi r^2 h$ given $h^2 + 4r^2 = 400$

Find r^2 in terms of h

$$4r^2 = 400 - h^2 \Rightarrow r^2 = 100 - \frac{1}{4}h^2$$

So $V = \pi r^2 h = \pi\left(100 - \frac{1}{4}h^2\right)h = \pi\left(100h - \frac{1}{4}h^3\right)$

Find the critical numbers:

$V'(h) = \pi\left(100 - \frac{3}{4}h^2\right)$ $\quad$ $V'(h)$ is always defined

$V'(h) = 0 \quad \Rightarrow \quad \pi\left(100 - \frac{3}{4}h^2\right) = 0$

$\Rightarrow \quad 100 = \frac{3}{4}h^2 \quad \Rightarrow \quad h^2 = \frac{400}{3}$

$h = \frac{20}{\sqrt{3}}$ or $h = \frac{-20}{\sqrt{3}}$

as a height we need $h > 0$.

Let's verify that $h = \frac{20}{\sqrt{3}}$ maximizes V.

$V''(h) = \pi\left(\frac{-6}{4}h\right) = \frac{-6\pi}{4}h$

$$V''\left(\frac{20}{\sqrt{3}}\right) = \frac{-6\pi}{4}\left(\frac{20}{\sqrt{3}}\right)$$ is negative.

Thus $h = \frac{20}{\sqrt{3}}$ maximizes V by the 2^{nd} der. test.

$V = \pi\left(100h - \frac{1}{4}h^3\right)$ so the maximum V is

$$V\left(\frac{20}{\sqrt{3}}\right) = \pi\left(100 \cdot \frac{20}{\sqrt{3}} - \frac{1}{4}\left(\frac{20}{\sqrt{3}}\right)^3\right)$$

$$= \pi\left(\frac{2000}{\sqrt{3}} - \frac{2000}{3\sqrt{3}}\right) = \frac{2000\pi}{\sqrt{3}}\left(1 - \frac{1}{3}\right)$$

$$= \frac{2000\pi}{\sqrt{3}}\left(\frac{2}{3}\right) = \frac{4000}{3\sqrt{3}}\pi$$

The maximum volume is $\frac{4000\pi}{3\sqrt{3}}$ cubic units.