Nov. 2 Math 1190 sec. 52 Fall 2016

Section 4.7: Optimization

Optimization problems arise in every field of study and every industry.

- minimize cost and maximize revenue,
- maximize crop yield,
- minimize driving time,
- maximize volume,
- minimize energy

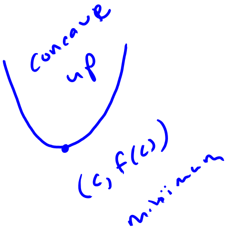
This will reduce to finding the extremum of some function f called the objective.

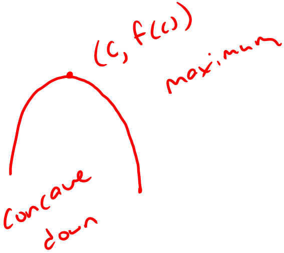
Often, some constraint (extra condition) must simultaneously be satisfied.

2^{nd} Derivative Test

Recall: If c is a critical number of the function f, and

- $f''(c) > 0$, then f has a local minumum at c
- $f''(c) < 0$, then f has a local maximum at c.





Applied Optimization Example

A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of its sides. What dimensions of the outer rectangle will require the smallest total length of fencing and how much fencing will be needed?

We want to minimize the length of fence.

We have to satisfy the constraint that the enclosure is 216 m^2.

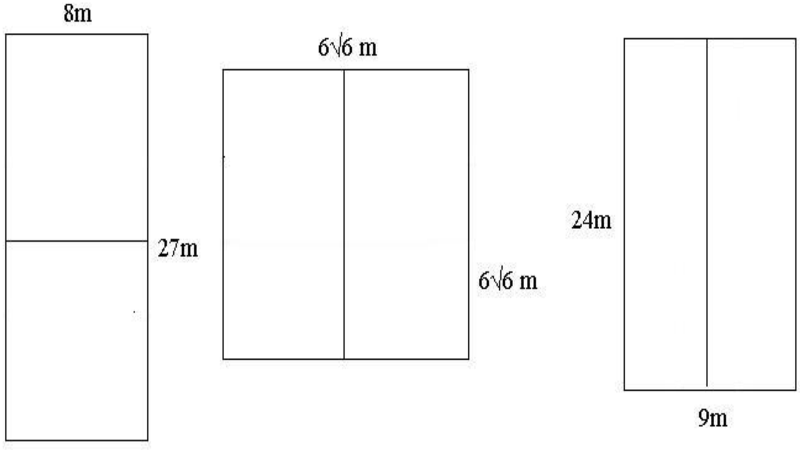
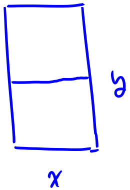


Figure: Different pea patch configuration that all enclose $216m^2$.

We consider a representative rectangular patch



with width x and length y meters.

The area $A = xy$ and the total perimeter $P = 3x + 2y$. We're given the constraint $A = 216\ m^2$ i.e. $xy = 216\ m^2$.

We want to minimize P. We need to reduce P to a function of only one variable. We can write y in terms of x by using $xy = 216$.

$y = \frac{216}{x}$ so $P = 3x + 2\left(\frac{216}{x}\right) = 3x + \frac{432}{x}$

$= 3x + 432x^{-1}$

Find critical numbers of P:

$P'(x) = 3 - 432x^{-2} = 3 - \frac{432}{x^2}$

$P'(x)$ is undefined if $x = 0$. This is not in the domain of P

Plus as a length, we need $x > 0$.

$P'(x) = 0 \Rightarrow 3 - \frac{432}{x^2} = 0 \Rightarrow 3 = \frac{432}{x^2}$

$x^2 = \frac{432}{3} = 144$

$\Rightarrow$ $x = 12$ or $x = -12$. As $x > 0$, we get one

critical number 12.

Let's verify that $x = 12$ actually minimizes P.

$$P'(x) = 3 - \frac{432}{x^2} = 3 - 432x^{-2}$$

2nd Der. test : $P''(x) = -2(-432)x^{-3} = \dfrac{2(432)}{x^3}$

$$P''(12) = \frac{2(432)}{(12)^3}$$ which is positive.

So $x = 12$ minimizes P by the 2nd der. test.

As $y = \frac{216}{x}$, the minimizing $y = \frac{216}{12} = 18$

when $x = 12$ and $y = 18$

$$P = 3(12) + 2(18) = 36 + 36 = 72$$

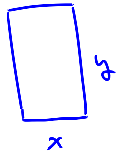
The optimum outer rectangle should be 12m x 18m

requiring 72 m of fencing.

Applied Optimization Example

Show that among all rectangles with perimeter 8m, the one with the largest area is a square.

Consider the rectangle with width x m and length y m. The area



$A = xy$ and the perimeter

$P = 2x + 2y$.

We want to maximize $A = xy$ given the constraint

$P = 8\text{m} = 2x + 2y$.

* For a square, it would turn out that $x = y$.

From $P = 8$, $2x = 8 - 2y \Rightarrow x = 4 - y$

So $A = xy = (4-y)y = 4y - y^2$

Find critical number(s):

$A'(y) = 4 - 2y$ this is always defined.

$A'(y) = 0 \Rightarrow 4 - 2y = 0 \Rightarrow 2y = 4 \Rightarrow y = 2$.

Let's verify that 2 maximizes A.

$A''(y) = -2$

so $A''(2) = -2$ which is negative.

Hence $y = 2$ maximizes A by the 2^{nd} derivative test.

As $x = 4 - y$, when $y = 2$ m, $x = 4 - 2 = 2$ m

The rectangle with the maximum area is the 2m $\times$ 2m square.

Applied Optimization Example

Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius 10.

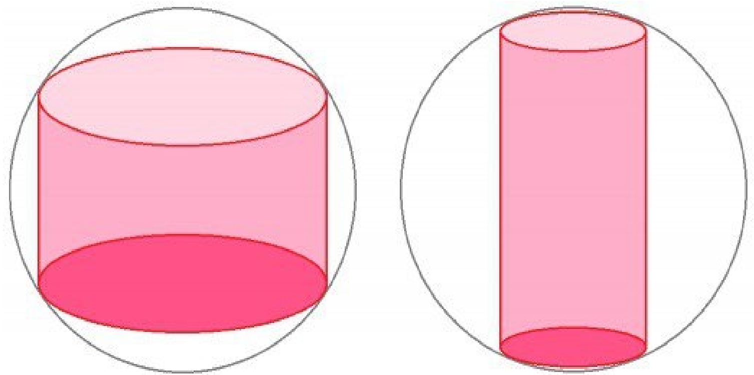
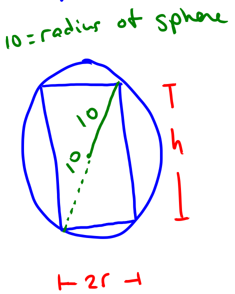


Figure: Different cylinders can be inscribed in the same sphere. We want to find the largest possible one.

We'll take a cross section of the sphere perpendicular to the base of the cylinder.



The volume V of a cylinder of radius r and height h is

$$V = \pi r^2 h$$

From the figure $h^2 + (2r)^2 = (2 \cdot 10)^2$

This is a constraint. We want to maximize

$$V = \pi r^2 h$$

We have $h^2 + 4r^2 = 400$. We can solve for r^2

$$4r^2 = 400 - h^2 \Rightarrow r^2 = 100 - \frac{1}{4}h^2$$

So $$V = \pi\left(100 - \frac{1}{4}h^2\right)h = \pi\left(100h - \frac{1}{4}h^3\right)$$

Find the critical number(s):

$$V'(h) = \pi\left(100 - \frac{3}{4}h^2\right)$$ which is always defined.

$$V'(h) = 0 \Rightarrow \pi\left(100 - \frac{3}{4}h^2\right) = 0$$

$$\Rightarrow \quad 100 - \frac{3}{4}h^2 = 0 \quad \Rightarrow \quad \frac{3}{4}h^2 = 100$$

$$h^2 = \frac{400}{3} \quad \Rightarrow \quad h = \frac{20}{\sqrt{3}} \text{ or } h = \frac{-20}{\sqrt{3}}$$

$h > 0$ as a height so we have one critical number $\frac{20}{\sqrt{3}}$.

Let's verify it maximizes V.

$$V'(h) = \pi\left(100 - \frac{3}{4}h^2\right)$$

$$V''(h) = \pi\left(\frac{-6}{4}h\right) = \frac{-6\pi}{4}h$$

$$\Rightarrow V''\left(\frac{20}{\sqrt{3}}\right) = \frac{-6\pi}{4}\left(\frac{20}{\sqrt{3}}\right)$$ which is negative.

So $h = \frac{20}{\sqrt{3}}$ maximizes the Volume.

$$V = \pi\left(100h - \frac{1}{4}h^3\right)$$

So the maximum Volume is

$$V\left(\frac{20}{\sqrt{3}}\right) = \pi\left(100\cdot\frac{20}{\sqrt{3}} - \frac{1}{4}\left(\frac{20}{\sqrt{3}}\right)^3\right)$$

$$= \pi \left(\frac{2000}{\sqrt{3}} - \frac{1}{4} \frac{20^3}{3\sqrt{3}} \right)$$

$$= \pi \left(\frac{2000}{\sqrt{3}} - \frac{2000}{3\sqrt{3}} \right)$$

$$= \frac{2000\pi}{\sqrt{3}} \left(1 - \frac{1}{3} \right) = \frac{2000\pi}{\sqrt{3}} \left(\frac{2}{3} \right)$$

The max volume is $\dfrac{4000\pi}{3\sqrt{3}}$.