November 2 Math 2306 sec 54 Fall 2015

Section 7.3: Translation Theorems

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\} = \frac{2}{s^3}$?

Consider the definition of $\mathscr{L}\left\{e^t t^2\right\} = \int_0^\infty e^{-st} \cdot e^t t^2 \, dt$

$$= \underbrace{\int_0^\infty e^{-(s-1)t} t^2 \, dt}_{\text{this is } \mathscr{L}\{t^2\} \text{ evaluated @ } s-1}$$

$$= \left. \frac{2}{s^3} \right|_{s-1} = \frac{2}{(s-1)^3}$$

Theorem (translation in s)

Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

$$\mathscr{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a)$$

For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}}.$$

Evaluate the Laplace Transform

(a) $\mathscr{L}\left\{e^{-3t}t^4\right\}$

$$= \frac{4!}{(s-(-3))^5} = \frac{4!}{(s+3)^5}$$

From the table

$$\mathscr{L}\{t^4\} = \frac{4!}{s^5}$$

(b) $\mathscr{L}\left\{e^{2t}\sin(\pi t)\right\}$

$$= \frac{\pi}{(s-2)^2 + \pi^2}$$

From the table

$$\mathscr{L}\{\sin \pi t\} = \frac{\pi}{s^2 + \pi^2}$$

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

s^2+2s+2 irreducible

complete the square

$s^2+2s+2 = s^2+2s+1-1+2$

$= (s+1)^2+1$

$\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\} = \mathscr{L}^{-1}\left\{\dfrac{s}{(s+1)^2+1}\right\}$

Use

$s = s+1-1$

$= \mathscr{L}^{-1}\left\{\dfrac{s+1-1}{(s+1)^2+1}\right\}$

$$= \mathcal{L}^{-1}\left\{ \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{s+1}{(s+1)^2+1} \right\} - \mathcal{L}^{-1}\left\{ \frac{1}{(s+1)^2+1} \right\}$$

$$= e^{-t}\cos t - e^{-t}\sin t$$

From the table $\mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1}$, $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

Do a partial fraction decomp

$$\frac{1+3s-s^2}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

Mult. both sides by $s(s-1)^2$

$$\underset{=}{-s^2} + \underset{=}{3s} + \underset{=}{1} = A(s-1)^2 + Bs(s-1) + Cs$$

$$= A(s^2-2s+1) + B(s^2-s) + Cs$$

$$= \underline{\underline{(A+B)}}s^2 + \underline{\underline{(-2A-B+C)}}s + \underline{\underline{A}}$$

$$A + B = -1 \qquad B = -1 - A = -2$$

$$-2A - B + C = 3 \qquad \text{add all three} \qquad C = 3$$

$$A = 1 \quad \Rightarrow A = 1$$

$$\mathcal{L}^{-1}\left\{\frac{1+3s-s^2}{s(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s-1} + \frac{3}{(s-1)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$= 1 - 2e^t + 3te^t$$

From the table

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

Solve the IVP using the Laplace Transform

Let $\mathcal{L}\{y(t)\} = Y(s)$

$$y''+4y'+4y = te^{-2t} \quad y(0)=1,\, y'(0)=0$$

$$\mathcal{L}\{y''+4y'+4y\} = \mathcal{L}\{te^{-2t}\}$$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \frac{1}{(s-(-2))^2}$$

$$s^2Y(s) - sy(0) - y'(0) + 4\left(sY(s) - y(0)\right) + 4Y(s) = \frac{1}{(s+2)^2}$$

$$(s^2+4s+4)Y(s) - s - 4 = \frac{1}{(s+2)^2}$$

$$(s+2)^2 Y(s) = \frac{1}{(s+2)^2} + s+4$$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{s+4}{(s+2)^2}$$

$$= \frac{1}{(s+2)^4} + \frac{s+2+2}{(s+2)^2}$$

$$= \frac{1}{(s+2)^4} + \frac{s+2}{(s+2)^2} + \frac{2}{(s+2)^2}$$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

From the table $\mathcal{L}\{t^3\} = \frac{3!}{s^4}$ and $\mathcal{L}\{t\} = \frac{1!}{s^2}$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^4} + \frac{1}{s+2} + \frac{2}{(s+2)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$= \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{(s+2)^4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$y = \frac{1}{6}t^3 e^{-2t} + e^{-2t} + 2te^{-2t}$$

Solve the IVP using the Laplace Transform

$y'' - 4y' + 13y = 39 \quad y(0) = 3, y'(0) = 1$

$$\mathcal{L}\{y'' - 4y' + 13y\} = \mathcal{L}\{39\} = 39\,\mathcal{L}\{1\}$$

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 13\mathcal{L}\{y\} = \frac{39}{s}$$

$$s^2 Y(s) - sy(0) - y'(0) - 4\left(sY(s) - y(0)\right) + 13Y(s) = \frac{39}{s}$$

$$(s^2 - 4s + 13)Y(s) - 3s - 1 + 12 = \frac{39}{s}$$

$$(s^2-4s+13)Y(s) = \frac{39}{s} + 3s - 11 = \frac{3s^2-11s+39}{s}$$

$$Y(s) = \frac{3s^2-11s+39}{s(s^2-4s+13)}$$

To do a decomp, note $s^2-4s+13$ is irreducible

$$\frac{3s^2-11s+39}{s(s^2-4s+13)} = \frac{A}{s} + \frac{Bs+C}{s^2-4s+13}$$

clear fractions

$$3s^2 - 11s + 39 = A(s^2 - 4s + 13) + (Bs + C)s$$

$$= As^2 - 4As + 13A + Bs^2 + Cs$$

$$= (A+B)s^2 + (-4A + C)s + 13A$$

$A + B = 3$ $\qquad$ $B = 3 - A = 0$

$-4A \quad + C = -11$ $\qquad$ $C = -11 + 4A = -11 + 12 = 1$

$13A \quad = 39 \implies A = 3$

$$Y(s) = \frac{3}{s} + \frac{1}{s^2-4s+13}$$

Complete the square

$$s^2-4s+13 = s^2-4s+4+9$$
$$= (s-2)^2+9$$

$$= \frac{3}{s} + \frac{1}{(s-2)^2+9}$$

$$y = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{3}{s} + \frac{1}{(s-2)^2+9}\right\}$$

$$= 3\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{(s-2)^2+9}\right\}$$

$$y = 3 + \frac{1}{3} e^{2t} \sin 3t$$

From the table

$$\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$$