Section 6.7 Inner Product Spaces

**Definition:** An **inner product** on a vector space $V$ is a function which assigns to each pair of vectors $u$ and $v$ in $V$ a real number denoted by $\langle u, v \rangle$ and that satisfies the following four axioms: For every $u, v, w$ in $V$ and scalar $c$

i. $\langle u, v \rangle = \langle v, u \rangle$,

ii. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$,

iii. $\langle cu, v \rangle = c \langle u, v \rangle$,

iv. $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ if and only if $u = 0$.

A vector space with an inner product is called an **inner product space**.
Norm, Distance, and Orthogonality

**Norm:** The norm of a vector $\mathbf{v}$ is $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

**A Unit Vector:** is a vector whose norm is 1.

**Distance:** The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u} - \mathbf{v}\|$.

**Orthogonality:** Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$

**Orthogonal Projection:** The orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$ is the vector

$$\hat{\mathbf{v}} = \left( \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \right) \mathbf{u}.$$

**Pythagorean Theorem:** If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$
Example (using some calculus)
Consider the set $C[a, b]$ of continuous functions defined on the interval $[a, b]$. For $f$ and $g$ in $C[a, b]$

$$< f, g > = \int_a^b f(t)g(t) \, dt$$

defines an inner product on $C[a, b]$.

(a) Show that $f(x) = \sin x$ and $g(x) = \cos x$ are orthogonal on $[-\pi, \pi]$.

$$< f, g > = \int_{-\pi}^{\pi} f(x)g(x) \, dx$$

$$= \int_{-\pi}^{\pi} \sin x \cos x \, dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2} \sin(2x) \, dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2} \sin(2x) \, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \sin(2x) \, dx$$

$$= 0$$

$\sin 2\theta = 2\sin \theta \cos \theta$
\[
= \frac{-1}{\pi} \left. \cos(2x) \right|_{-\pi}^{\pi} = \frac{-1}{\pi} \left( \cos(2\pi) - \cos(-2\pi) \right)
\]
\[
= \frac{-1}{\pi} \left( 1 - 1 \right) = 0
\]

So \( f(x) \) and \( g(x) \) are orthogonal.
Example continued...

(b) Find the distance between $f$ and $g$ in $C[-\pi, \pi]$.

The square of the distance $\|f - g\|^2$

$$\|f - g\|^2 = \langle f - g, f - g \rangle = \int_{-\pi}^{\pi} (f(x) - g(x))(f(x) - g(x)) \, dx$$

$$= \int_{-\pi}^{\pi} (\sin^2 x - 2\sin x \cos x + \cos^2 x) \, dx$$

$$= \int_{-\pi}^{\pi} (\sin^2 x + \cos^2 x - 2\sin x \cos x) \, dx$$

$$= \int_{-\pi}^{\pi} (1 - 2\sin x \cos x) \, dx$$
\[
\int_{-\pi}^{\pi} dx - 2 \int_{-\pi}^{\pi} \sin x \cos x \, dx \\
= \int_{-\pi}^{\pi} dx = \pi \bigg|_{-\pi}^{\pi} = \pi - (-\pi) = 2\pi
\]

0° by part (a)

So the distance between \(\sin x\) and \(\cos x\)

is \(\sqrt{2\pi}\).
Some Key Inequalities

**Cauchy Schwarz:** In an inner product space $V$, for any $u, v$ we have

$$|\langle u, v \rangle| \leq \|u\| \|v\|.$$ 

**Triangle:** For any $u, v$

$$\|u + v\| \leq \|u\| + \|v\|.$$
Gram Schmidt

Let $\mathbb{P}_2[-1, 1]$ be the set of polynomials of degree at most 2 defined for $-1 \leq t \leq 1$. Using the following inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(t)q(t) \, dt$$

find an orthogonal basis for $\mathbb{P}_2$ by starting with the elementary basis $\{1, t, t^2\}$.

Well call the new basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$\vec{v}_1 = 1$$

$$\vec{v}_2 = t - \frac{\langle t, 1 \rangle}{\langle 1, 1 \rangle} 1$$

$$\langle t, 1 \rangle = \int_{-1}^{1} t \cdot 1 \, dt = \frac{t^2}{2} \bigg|_{-1}^{1} = \frac{1^2}{2} - \frac{(-1)^2}{2} = \frac{1}{2} - \frac{1}{2} = 0$$
So \( \vec{v}_1 = 1 \) and \( \vec{v}_2 = t \)

\[
\vec{v}_3 = t^2 - \frac{\langle t^2, 1 \rangle}{\langle 1, 1 \rangle} 1 - \frac{\langle t^2, t \rangle}{\langle t, t \rangle} t
\]

\[
\langle t^2, 1 \rangle = \int_1^1 t^2 \, dt = \frac{t^3}{3} \bigg|_1^1 = \frac{t^3}{3} - \frac{(-1)^3}{3} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}
\]

\[
\langle 1, 1 \rangle = \int_1^1 1 \, dt = \int_1^1 dt = t \bigg|_1^1 = 1 - (-1) = 2
\]

\[
\langle t^2, t \rangle = \int_1^1 t^2 \cdot t \, dt = \int_1^1 t^3 \, dt = \frac{t^4}{4} \bigg|_1^1 = \frac{t^4}{4} - \frac{(-1)^4}{4} = \frac{1}{4} - \frac{1}{4} = 0
\]
\[ \vec{V}_3 = t^2 - \frac{2\sqrt{3}}{2} \cdot 1 = t^2 - \frac{1}{3} \]

An orthogonal basis is

\[ \{ 1, t, t^2 - \frac{1}{3} \} \]
Section 5.1: Eigenvectors and Eigenvalues

Consider the matrix $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ and the vectors $u = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Plot $u$, $Au$, $v$, and $Av$ on the axis on the next slide.

$$A\hat{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 - 2 \\ -1 + 0 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

$$A\hat{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 - 2 \\ 2 + 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$
Example Plot
Eigenvalues and Eigenvectors

Note that in this example, the matrix $A$ seems to both stretch and rotate the vector $u$. But the action of $A$ on the vector $v$ is just a stretch/compress.

We wish to consider matrices with vectors that satisfy relationships such as

$$Ax = 2x,$$  
$$Ax = -4x,$$  
or more generally  
$$Ax = \lambda x$$

for constant $\lambda$—and for nonzero vector $x$. 
Definition of Eigenvector and Eigenvalue

**Definition:** Let $A$ be an $n \times n$ matrix. A nonzero vector $x$ such that

$$Ax = \lambda x$$

for some scalar $\lambda$ is called an **eigenvector** of the matrix $A$.

A scalar $\lambda$ such that there exists a nonzero vector $x$ satisfying $Ax = \lambda x$ is called an **eigenvalue** of the matrix $A$. Such a nonzero vector $x$ is an **eigenvector corresponding to** $\lambda$.

Note that built right into this definition is that the eigenvector $x$ must be nonzero!
Example

The number \( \lambda = -4 \) is an eigenvalue of the matrix
\[
A = \begin{bmatrix}
1 & 6 \\
5 & 2
\end{bmatrix}
\]. Find the corresponding eigenvectors.

We want \( \vec{x} \neq \vec{0} \) such that \( A\vec{x} = -4\vec{x} \).

For \( \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) our equation is
\[
\begin{bmatrix}
1 & 6 \\
5 & 2
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -4 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

as a system
\[
\begin{align*}
x_1 + 6x_2 &= -4x_1 \\
5x_1 + 2x_2 &= -4x_2
\end{align*}
\]

we can subtract the stuff on the right
\[
(1 - (-4))x_1 + 6x_2 = 0 \quad \Rightarrow \quad \begin{bmatrix} 5 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
\[
\begin{bmatrix}
5 & 6 & 0 \\
5 & 6 & 0 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 6/5 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]
\[x_1 = -\frac{6}{5} x_2\]
\[x_2 \text{ - free}\]

so all eigen vectors look like

\[
\begin{bmatrix}
\frac{-6}{5} \\
1 \\
\end{bmatrix}
\]

\(\chi = \chi_2\begin{bmatrix}
\frac{-6}{5} \\
1 \\
\end{bmatrix}\) for \(x_2 \neq 0\).
Eigenspace

**Definition:** Let $A$ be an $n \times n$ matrix and $\lambda$ and eigenvalue of $A$. The set of all eigenvectors corresponding to $\lambda$—i.e. the set

$$\{x \in \mathbb{R}^n \mid x \neq 0 \text{ and } Ax = \lambda x\},$$

is called the **eigenspace of $A$ corresponding to $\lambda$**.

**Remark:** The eigenspace is the same as the null space of the matrix $A - \lambda I$. It follows that the eigenspace is a subspace of $\mathbb{R}^n$.

In the $2 \times 2$ case

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$
Example

The matrix $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ has eigenvalue $\lambda = 2$. Find a basis for the eigenspace of $A$ corresponding to $\lambda$.

The homogeneous system has coeff. matrix $A - 2I$

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & \frac{1}{2} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad x_1 = \frac{1}{2}x_2 - 3x_3$$

$x_2, x_3$ are free.
Solutions

\[ x = x_2 \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \end{bmatrix} \]

A basis for the eigen space is

\[ \left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \end{bmatrix} \right\} \]
Matrices with Nice Structure

**Theorem:** If $A$ is an $n \times n$ triangular matrix, then the eigenvalues of $A$ are its diagonal elements.

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & \pi & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

A is lower triangular. The eigenvalues are

$$\lambda_1 = 3, \quad \lambda_2 = \pi, \quad \lambda_3 = 1$$
Example
Suppose \( \lambda = 0 \) is an eigenvalue\(^2\) of a matrix \( A \). Argue that \( A \) is not invertible.

Then must be an eigenvector \( \vec{x} \neq \vec{0} \) such that

\[
A\vec{x} = 0 \Rightarrow \vec{x} = \vec{0}.
\]

Since \( \vec{x} \neq \vec{0} \) we have a nontrivial solution to the homogeneous equation \( A\vec{x} = \vec{0} \).

By the "invertible matrix theorem," \( A \) is not invertible.

\(^2\)Eigenvectors must be nonzero vectors, but it is perfectly legitimate to have a zero eigenvalue!
Theorems

**Theorem:** A square matrix $A$ is invertible if and only if zero is **not** an eigenvalue.

**Theorem:** If $v_1, \ldots, v_p$ are eigenvectors of a matrix $A$ corresponding to distinct eigenvalues, $\lambda_1, \ldots, \lambda_p$, then the set $\{v_1, \ldots, v_p\}$ is linearly independent.