November 2 Math 3260 sec. 58 Fall 2017

Section 6.7 Inner Product Spaces

Definition: An **inner product** on a vector space V is a function which assigns to each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V a real number denoted by $<\mathbf{u},\mathbf{v}>$ and that satisfies the following four axioms: For every $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and scalar c

i $<\mathbf{u},\mathbf{v}>=<\mathbf{v},\mathbf{u}>$,

ii $<\mathbf{u}+\mathbf{v},\mathbf{w}>=<\mathbf{u},\mathbf{w}>+<\mathbf{v},\mathbf{w}>$,

iii $<c\mathbf{u},\mathbf{v}>=c<\mathbf{u},\mathbf{v}>$,

iv $<\mathbf{u},\mathbf{u}>\geq 0$ and $<\mathbf{u},\mathbf{u}>=0$ if and only if $\mathbf{u}=\mathbf{0}$.

A vector space with an inner product is called an **inner product space.**

Norm, Distance, and Orthogonality

Norm: The norm of a vector $\mathbf{v}$ is $\|\mathbf{v}\| = \sqrt{<\mathbf{v},\mathbf{v}>}$.

A Unit Vector: is a vector whose norm is 1.

Distance: The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u}-\mathbf{v}\|$.

Orthogonality: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $<\mathbf{u},\mathbf{v}>= 0$

Orthogonal Projection: The orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$ is the vector

$$\hat{\mathbf{v}} = \left(\frac{<\mathbf{v},\mathbf{u}>}{<\mathbf{u},\mathbf{u}>}\right)\mathbf{u}.$$

Pythagorean Theorem: If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Example (using some calculus)

Consider the set $C[a, b]$ of continuous functions defined on the interval $[a, b]$. For f and g in $C[a, b]$

$$< f, g >= \int_a^b f(t)g(t)\, dt$$

defines an inner product on $C[a, b]$.

(a) Show that $f(x) = \sin x$ and $g(x) = \cos x$ are orthogonal on $[-\pi, \pi]$.

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

$$= \int_{-\pi}^{\pi} \sin x \cos x \, dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2} \sin(2x)\, dx$$

ID

$\sin 2\theta = 2 \sin\theta \cos\theta$

$$= \frac{-1}{4}\cos(2x)\Big|_{-\pi}^{\pi} = \frac{-1}{4}\left(\cos(2\pi) - \cos(-2\pi)\right)$$

$$= \frac{-1}{4}(1-1) = 0$$

So $f(x)$ and $g(x)$ are orthogonal.

Example continued...

(b) Find the distance between *f* and *g* in $C[-\pi, \pi]$.

The square of the distance $\|f-g\|^2$

$$\|f-g\|^2 = \langle f-g, f-g \rangle = \int_{-\pi}^{\pi} (f(x)-g(x))(f(x)-g(x))\,dx$$

$$= \int_{-\pi}^{\pi} (\sin^2 x - 2\sin x\cos x + \cos^2 x)\,dx$$

$$= \int_{-\pi}^{\pi} (\sin^2 x + \cos^2 x - 2\sin x\cos x)\,dx$$

$$= \int_{-\pi}^{\pi} (1 - 2\sin x\cos x)\,dx$$

$$= \int_{-\pi}^{\pi} dx \quad - 2 \int_{-\pi}^{\pi} \sin x \cos x \, dx$$

"0" by part (a)

$$= \int_{-\pi}^{\pi} dx = x \Big|_{-\pi}^{\pi} = \pi - (-\pi) = 2\pi$$

So the distance between $\sin x$ and $\cos x$ is $\sqrt{2\pi}$.

Some Key Inequalities

Cauchy Schwarz: In an inner product space V, for any **u**, **v** we have

$$|<\mathbf{u},\mathbf{v}>| \leq \|\mathbf{u}\|\|\mathbf{v}\|.$$

Triangle: For any **u**, **v**

$$\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Gram Schmidt

Let $\mathbb{P}_2[-1,1]$ be the set of polynomials of degree at most 2 defined for $-1 \le t \le 1$. Using the following inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} \mathbf{p}(t)\mathbf{q}(t)\, dt$$

find an orthogonal basis for $\mathbb{P}_2$ by starting with the elementary basis $\{1, t, t^2\}$.

We'll call the new basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$\vec{v}_1 = 1$$

$$\vec{v}_2 = t - \frac{\langle t, 1\rangle}{\langle 1, 1\rangle} 1$$

$$\langle t, 1\rangle = \int_{-1}^{1} t \cdot 1\, dt = \left.\frac{t^2}{2}\right|_{-1}^{1} = \frac{1^2}{2} - \frac{(-1)^2}{2} = \frac{1}{2} - \frac{1}{2} = 0$$

So $\vec{v}_1 = 1$ and $\vec{v}_2 = t$

$$\vec{v}_3 = t^2 - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle} 1 - \frac{\langle t^2, t\rangle}{\langle t, t\rangle} t$$

$$\langle t^2, 1\rangle = \int_{-1}^{1} t^2 \cdot 1 \, dt = \frac{t^3}{3}\Big|_{-1}^{1} = \frac{1^3}{3} - \frac{(-1)^3}{3} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\langle 1, 1\rangle = \int_{-1}^{1} 1 \cdot 1 \, dt = \int_{-1}^{1} dt = t\Big|_{-1}^{1} = 1 - (-1) = 2$$

$$\langle t^2, t\rangle = \int_{-1}^{1} t^2 \cdot t \, dt = \int_{-1}^{1} t^3 \, dt = \frac{t^4}{4}\Big|_{-1}^{1} = \frac{1^4}{4} - \frac{(-1)^4}{4} = \frac{1}{4} - \frac{1}{4} = 0$$

so $\vec{V}_3 = t^2 - \frac{2/3}{2} \cdot 1 = t^2 - \frac{1}{3}$

An orthogonal basis is

$$\left\{1, t, t^2 - \frac{1}{3}\right\}$$

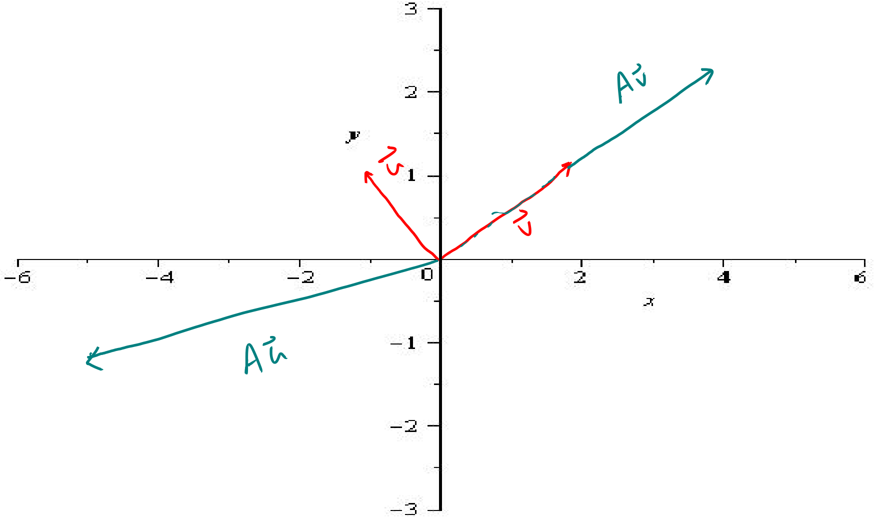
Section 5.1: Eigenvectors and Eigenvalues

Consider the matrix $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ and the vectors $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Plot $\mathbf{u}$, $A\mathbf{u}$, $\mathbf{v}$, and $A\mathbf{v}$ on the axis on the next slide.

$$A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3-2 \\ -1+0 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6-2 \\ 2+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Example Plot



Eigenvalues and Eigenvectors

Note that in this example, the matrix A seems to both stretch and rotate the vector $\mathbf{u}$. But the *action of* A on the vector $\mathbf{v}$ is just a stretch/compress.

We wish to consider matrices with vectors that satisfy relationships such as

$$A\mathbf{x} = 2\mathbf{x}, \quad \text{or} \quad A\mathbf{x} = -4\mathbf{x}, \quad \text{or more generally} \quad A\mathbf{x} = \lambda\mathbf{x}$$

for constant λ—and for nonzero vector $\mathbf{x}$.

Definition of Eigenvector and Eigenvalue

Definition: Let A be an $n \times n$ matrix. A nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ is called an **eigenvector** of the matrix A.

A scalar λ such that there exists a nonzero vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is called an **eigenvalue** of the matrix A. Such a nonzero vector $\mathbf{x}$ is an *eigenvector corresponding to* λ.

Note that built right into this definition is that the eigenvector **$\mathbf{x}$ must be** nonzero!

Example

The number $\lambda = -4$ is an eigenvalue of the matrix matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. Find the corresponding eigenvectors.

We want $\vec{x} \neq \vec{0}$ such that $A\vec{x} = -4\vec{x}$

For $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ our equation is $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -4\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

as a system

$$x_1 + 6x_2 = -4x_1$$
$$5x_1 + 2x_2 = -4x_2$$

we can subtract the stuff on the right

$$\left.\begin{aligned} (1-(-4))x_1 + 6x_2 &= 0 \\ 5x_1 + (2-(-4))x_2 &= 0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 6 & 0 \\ 5 & 6 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 6/5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} x_1 = -\frac{6}{5} x_2 \\ x_2 \text{ - free} \end{array}$$

So all eigenvectors look like

$$\vec{x} = x_2 \begin{bmatrix} -\frac{6}{5} \\ 1 \end{bmatrix} \quad \text{for } x_2 \neq 0.$$

Eigenspace

Definition: Let A be an $n \times n$ matrix and λ and eigenvalue of A. The set of all eigenvectors corresponding to λ—i.e. the set

$$\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \neq \mathbf{0} \text{ and } A\mathbf{x} = \lambda\mathbf{x}\},$$

is called the **eigenspace of A corresponding to λ.**

Remark: The eigenspace is the same as the null space of the matrix $A - \lambda I$. It follows that the eigenspace is a subspace of $\mathbb{R}^n$.

In the 2x2 case

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

Example

The matrix $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ has eigenvalue $\lambda = 2$. Find a basis for the eigenspace of A corresponding to λ.

The homogeneous system has coef. matrix $A - 2I$

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = \frac{1}{2}x_2 - 3x_3$

x_2, x_3 - free

Solutions

$$\vec{x} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

A basis for the eigen space is

$$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Matrices with Nice Structure

Theorem: If A is an $n \times n$ triangular matrix, then the eigenvalues of A are its diagonal elements.

Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & \pi & 0 \\ -1 & 0 & 1 \end{bmatrix}$

A is lower triangular. The eigen values are

$\lambda_1 = 3, \quad \lambda_2 = \pi, \quad \lambda_3 = 1$

Example

Suppose $\lambda = 0$ is an eigenvalue[2] of a matrix A. Argue that A is not invertible.

Then must be an eigenvector $\vec{x} \neq \vec{0}$ such that

$A\vec{x} = 0\vec{x} = \vec{0}$. Since $\vec{x} \neq \vec{0}$ we have a nontrivial

solution to the homogeneous equation $A\vec{x} = \vec{0}$.

By the "invertible matrix theorem" A is

not invertible.

[2] Eigenvectors must be nonzero vectors, but it is perfectly legitimate to have a zero eigenvalue!

Theorems

Theorem: A square matrix A is invertible if and only if zero is **not** and eigenvalue.

Theorem: If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are eigenvectors of a matrix A corresponding to distinct eigenvalues, $\lambda_1, \ldots, \lambda_p$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent.