November 30 Math 2306 sec 51 Fall 2015

Section 11.3: Fourier Cosine and Sine Series

Half Range Sine and Half Range Cosine Series: For f defined on $0 < x < p$.

Half range cosine series $\quad f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$

where $\quad a_0 = \dfrac{2}{p}\displaystyle\int_0^p f(x)\,dx \quad$ and $\quad a_n = \dfrac{2}{p}\displaystyle\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$

Half range sine series $\quad f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$

where $\quad b_n = \dfrac{2}{p}\displaystyle\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$

Half Range Series

For the given function, plot the graph of the function along with three full periods on the interval $(-3p, 3p)$ of (a) the half range cosine series and (b) the half range sine series.

$$f(x) = \begin{cases} x, & 0 \le x < \frac{3}{2} \\ 3 - x, & \frac{3}{2} \le x < 3 \end{cases} \qquad p = 3$$

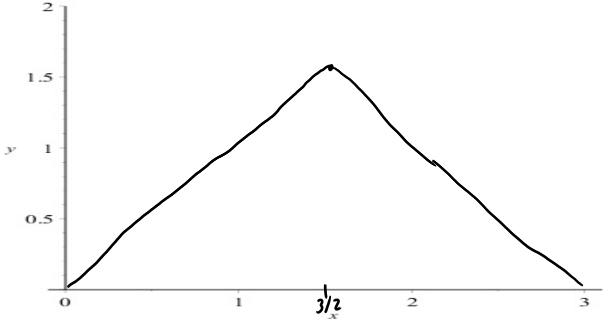

Figure: Plot of f alone.

(a) Even Extension

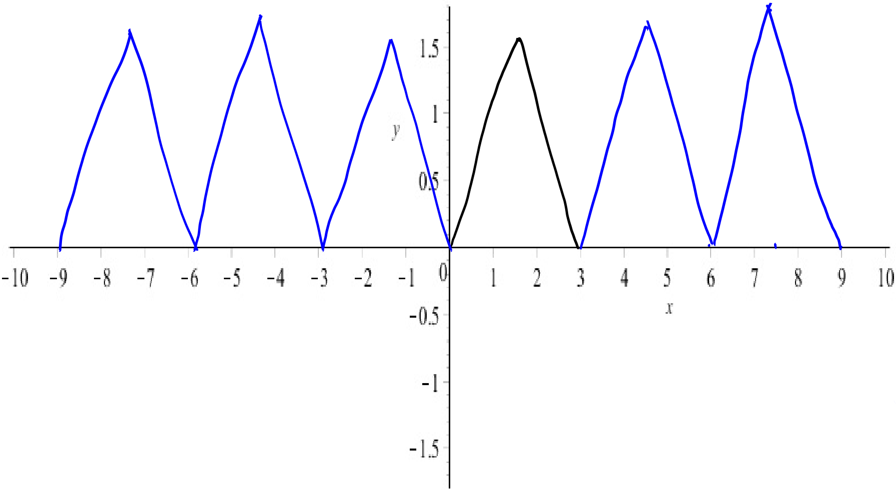

Figure: Half range co sine series.

(b) Odd Extension

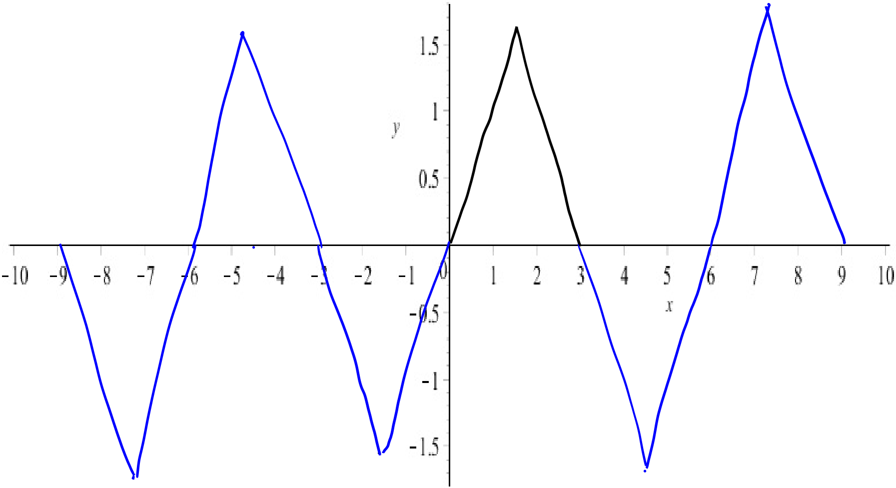

Figure: Half range sine series.

Solution of a Differential Equation

An undamped spring mass system has a mass of 2 kg attached to a spring with spring constant 128 N/m. The mass is driven by an external force $f(t) = 2t$ for $-1 < t < 1$ that is 2-periodic so that $f(t+2) = f(t)$ for all $t > 0$. Determine a particular solution x_p for the displacement for $t > 0$.

$$mx'' + kx = f(t) \qquad m=2, \quad k=128$$

$$2x'' + 128x = f(t) \quad \Rightarrow \quad x'' + 64x = \frac{1}{2}f(t)$$

Since f is odd and 2-periodic, we can express f in a sine series

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t)$$

where $$b_n = \frac{2}{1}\int_0^1 f(t)\,\sin(n\pi t)\,dt$$

$$= 2\int_0^1 2t\,\sin(n\pi t)\,dt$$

Parts:

$u = t \qquad du = dt$

$dv = \sin(n\pi t)\,dt$

$v = \frac{-1}{n\pi}\cos(n\pi t)$

$$= 4\left[\frac{-t}{n\pi}\cos(n\pi t)\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi t)\,dt\right.$$

$$= 4\left[\frac{-1}{n\pi}\cos(n\pi) - 0 + \frac{1}{n^2\pi^2}\underbrace{\sin(n\pi t)\Big|_0^1}_{0}\right.$$

$$= 4\left(\frac{-1}{n\pi}\right)(-1)^n = \frac{4(-1)^{n+1}}{n\pi}$$

$$f(t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

So the ODE becomes

$$x'' + 64x = \frac{1}{2} \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

$$x'' + 64x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

In the spirit of the method of undetemined coeff.

let's guess that

$$x_p = \sum_{n=1}^{\infty} B_n \sin(n\pi t)$$

Substitute this into the D.E.

$$x_p' = \sum_{n=1}^{\infty} B_n (n\pi) \cos(n\pi t)$$

$$x_p'' = \sum_{n=1}^{\infty} B_n (n\pi)^2 (-\sin(n\pi t)) = -\sum_{n=1}^{\infty} B_n (n\pi)^2 \sin(n\pi t)$$

So $$x_p'' + 64 x_p = -\sum_{n=1}^{\infty} B_n (n\pi)^2 \sin(n\pi t) + 64 \sum_{n=1}^{\infty} B_n \sin(n\pi t)$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

Collecting on the left

$$\sum_{n=1}^{\infty}\left[B_n \sin(n\pi t)\left(-(n\pi)^2+64\right)\right] = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin(n\pi t)$$

$$\sum_{n=1}^{\infty}\left[(64-n^2\pi^2)B_n \sin(n\pi t)\right] = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin(n\pi t)$$

Matching coefficients of $\sin(n\pi t)$ for each n

$$(64-n^2\pi^2)B_n = \frac{2(-1)^{n+1}}{n\pi}$$

Note

$64-n^2\pi^2 \neq 0$

for all n

$$B_n = \frac{2(-1)^{n+1}}{n\pi\,(64-n^2\pi^2)}$$

Finally, the particula solution is

$$x_p = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi\,(64-n^2\pi^2)} \sin(n\pi t).$$

An Eigenvalue Problem

An *eigenvalue problem* consists of a Boundary Value Problem which includes an unknown parameter λ. The task is to determine values of the parameter and corresponding nonzero functions (i.e. nontrivial) that solve the BVP. The values are called eigenvalues and the corresponding functions are called eigenfunctions.

Example: Solve $-u'' = \lambda u$ for $0 < x < 1$ subject to $u(0) = 0$ and $u(1) = 0$.

Rewrite the DE $\quad u'' + \lambda u = 0$

The characteristic eqn is $\quad m^2 + \lambda = 0$

We can consider cases, $\lambda = 0$, $\lambda < 0$, or $\lambda > 0$

Case 1: $\lambda = 0$ $\quad m^2 = 0 \Rightarrow m = 0$ repeated

the solutions are $u_1 = e^{0x} = 1$, $u_2 = xe^{0x} = x$

The general soln. is $u = C_1 + C_2 x$

Apply $u(0) = 0$ and $u(1) = 0$

$$\left.\begin{aligned} u(0) &= C_1 + C_2 \cdot 0 = C_1 = 0 \\ u(1) &= C_2 \cdot 1 = 0 \Rightarrow C_2 = 0 \end{aligned}\right\} \Rightarrow u(x) = 0$$

only the trivial solution exists

So 0 is not an eigen value.

Case 2: $\lambda < 0$ let $\lambda = -\alpha^2$, for $\alpha > 0$

$$m^2 - \alpha^2 = 0 \Rightarrow m = \pm\alpha$$

$$u_1 = e^{\alpha x} \quad \text{and} \quad u_2 = e^{-\alpha x}$$

$$u = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

$$u(0) = c_1 e^0 + c_2 e^0 = 0 \Rightarrow c_1 + c_2 = 0$$

$$u(1) = c_1 e^{\alpha} + c_2 e^{-\alpha} = 0 \Rightarrow c_1 e^{\alpha} + c_2 e^{-\alpha} = 0$$

$$c_1 = -c_2$$

$$-c_2 e^{\alpha} + c_2 e^{-\alpha} = 0 \implies c_2 e^{\alpha} = c_2 e^{-\alpha}$$

$$c_2 e^{2\alpha} = c_2 \implies c_2 = 0 \quad \text{since } \alpha > 0$$

Since $c_1 = -c_2$, $c_1 = 0$

So the only solution is the trivial one

$u = 0$.

$\lambda < 0$ doesn't give any eigenvalues.

We'll wrap this up on

Wednesday.