November 30 Math 2306 sec 54 Fall 2015

Section 11.3: Fourier Cosine and Sine Series

Half Range Sine and Half Range Cosine Series: For f defined on $0 < x < p$.

Half range cosine series $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

where $$a_0 = \frac{2}{p}\int_0^p f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$$

Half range sine series $$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

where $$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$$

Solution of a Differential Equation

An undamped spring mass system has a mass of 2 kg attached to a spring with spring constant 128 N/m. The mass is driven by an external force $f(t) = 2t$ for $-1 < t < 1$ that is 2-periodic so that $f(t+2) = f(t)$ for all $t > 0$. Determine a particular solution x_p for the displacement for $t > 0$.

$$mx'' + kx = f(t) \qquad m = 2 \quad \text{and} \quad k = 128$$

$$2x'' + 128x = f(t) \implies x'' + 64x = \frac{1}{2}f(t)$$

f is odd and 2-periodic, so we can write

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t) \qquad \text{here } p = 1 \quad \frac{n\pi}{p} = n\pi$$

where $\quad b_n = \frac{2}{1}\int_0^1 f(t)\,\sin(n\pi t)\,dt$

$$= 2\int_0^1 2t\,\sin(n\pi t)\,dt$$

$$= 4\left[\frac{-t}{n\pi}\cos(n\pi t)\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi t)\,dt\right.$$

$$= 4\left[\frac{-1}{n\pi}\cos(n\pi) - 0 + \frac{1}{n^2\pi^2}\underbrace{\sin(n\pi t)\Big|_0^1}_{0}\right.$$

$$= 4\left(\frac{-1}{n\pi}\right)(-1)^n = \frac{4(-1)^{n+1}}{n\pi}$$

By parts

$u = t \quad du = dt$

$dv = \sin(n\pi t)\,dt$

$v = \frac{-1}{n\pi}\cos(n\pi t)$

So $f(t)=\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n\pi}\sin(n\pi t)$

The DE is

$$x''+64x=\frac{1}{2}\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n\pi}\sin(n\pi t)$$

$$=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin(n\pi t)$$

In the spirit of the method of undetermined coefficients, assume $x_p=\sum_{n=1}^{\infty}B_n\sin(n\pi t)$

Substitute x_p into the D.E.

$$x_p' = \sum_{n=1}^{\infty} B_n (n\pi) \cos(n\pi t)$$

$$x_p'' = \sum_{n=1}^{\infty} -B_n (n\pi)^2 \sin(n\pi t)$$

$$x_p'' + 64x_p = \sum_{n=1}^{\infty} -(n\pi)^2 B_n \sin(n\pi t) + 64 \sum_{n=1}^{\infty} B_n \sin(n\pi t)$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

Collecting on the left

$$\sum_{n=1}^{\infty}\left[\left(-(n\pi)^2+64\right)B_n\sin(n\pi t)\right]=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin(n\pi t)$$

Match term by term

$$\left(64-(n\pi)^2\right)B_n=\frac{2(-1)^{n+1}}{n\pi}$$

Note

$64-n^2\pi^2\neq 0$

for all

n

$$\Rightarrow\quad B_n=\frac{2(-1)^{n+1}}{n\pi\left(64-n^2\pi^2\right)}$$

The particular solution is

$$x_p = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi(64-n^2\pi^2)} \sin(n\pi t).$$

An Eigenvalue Problem

An *eigenvalue problem* consists of a Boundary Value Problem which includes an unknown parameter λ. The task is to determine values of the parameter and corresponding nonzero functions (i.e. nontrivial) that solve the BVP. The values are called eigenvalues and the corresponding functions are called eigenfunctions.

Example: Solve $-u'' = \lambda\, u$ for $0 < x < 1$ subject to $u(0) = 0$ and $u(1) = 0$.

We can write the DE as

$$u'' + \lambda u = 0 \quad \text{with} \quad u(0) = 0, \; u(1) = 0$$

The characteristic equation is

$$m^2 + \lambda = 0$$

We'll consider three cases $\lambda = 0$, $\lambda < 0$ and $\lambda > 0$

Case 1: $\lambda = 0$ $\qquad m^2 = 0 \qquad m = 0$ repeated root

$$u_1 = e^{0x} = 1 \quad \text{and} \quad u_2 = xe^{0x} = x$$

The general solution is $u = c_1 + c_2 x$

Impose $u(0) = 0$ and $u(1) = 0$

$$u(0) = c_1 + c_2 \cdot 0 = 0 \quad \Rightarrow \quad c_1 = 0$$
$$u(1) = c_2 \cdot 1 = 0 \quad \Rightarrow \quad c_2 = 0$$

$\lambda = 0$ only gives the trivial solution $u(x) = 0$.

Zero is <u>not</u> an eigenvalue.

Case 2: $\lambda < 0$, let $\lambda = -\alpha^2$ for $\alpha > 0$

Then $m^2 - \alpha^2 = 0 \Rightarrow m = \pm\alpha$

$$u_1 = e^{\alpha x} \quad \text{and} \quad u_2 = e^{-\alpha x}$$

$$u = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

Apply $u(0)=0$ and $u(1)=0$

$$u(0) = c_1 e^0 + c_2 e^0 = 0 \quad \Rightarrow \quad c_1 + c_2 = 0$$

$$u(1) = c_1 e^{\alpha} + c_2 e^{-\alpha} = 0 \quad \Rightarrow \quad c_1 e^{\alpha} + c_2 e^{-\alpha} = 0$$

$$c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1$$

$$c_1 e^{\alpha} - c_1 e^{-\alpha} = 0 \quad \Rightarrow \quad c_1 (e^{\alpha} - e^{-\alpha}) = 0$$

$$c_1 e^{\alpha} = c_1 e^{-\alpha}$$

$$\Rightarrow \quad c_1 e^{2\alpha} = c_1$$

either $e^{2\alpha}=1$ or $c_1=0$

Since $\alpha>0$, $e^{2\alpha}\neq 1$ so $c_1=0$.

$c_2=-c_1=0$ so again we only get the trivial solution $u(x)=0$.

No eigenvalues are negative.

Case 3: $\lambda>0$ Let $\lambda=\alpha^2$ for $\alpha>0$

Then $m^2+\alpha^2=0 \implies m=\pm i\alpha$

$$u_1 = \cos(\alpha x) \qquad u_2 = \sin(\alpha x)$$

So $$u(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$$

Apply $u(0)=0$ and $u(1)=0$

$$u(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 = 0$$

$$u(1) = c_2 \sin(\alpha) = 0$$

either $c_2 = 0$ (not interesting)

or $\quad \sin(\alpha) = 0$

There are infinitely many solutions

$$\alpha = n\pi \quad \text{for } n = 1, 2, 3, \ldots$$

The eigen values are

$$\lambda_n = \alpha_n^2 = n^2\pi^2$$

The associated eigen functions are $u_n = \sin(n\pi x)$.

A Related Eigenvalue Problem

It can be shown that the problem

$$-u'' = \lambda\, u \quad \text{for} \quad 0 < x < 1 \quad \text{subject to} \quad u'(0) = u'(1) = 0$$

has eigenvalues and eigen functions

$$\lambda_n = n^2\pi^2 \quad n \geq 0, \quad u_0 = 1, \quad u_n(x) = \cos(n\pi x), \quad n \geq 1$$

Another Related Eigenvalue Problem

It can be shown that the problem

$$-u'' = \lambda\, u \text{ for } 0 < x < 1 \quad \text{subject to} \quad u(0) = u(1) \text{ and } u'(0) = u'(1)$$

has eigenvalues and eigen functions

$$\lambda_n = n^2\pi^2 \quad n \geq 0, \quad u_0(x) = 1, \quad u_n(x) = a_n \cos(n\pi x) + b_n \sin(n\pi x)$$