November 30 Math 2306 sec. 57 Fall 2017

Section 18: Sine and Cosine Series

If f is even on $(-p, p)$, then the Fourier series of f has only constant and cosine terms. Moreover

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

where

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx$$

and

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$$

Fourier Series of an Odd Function

If f is odd on $(-p, p)$, then the Fourier series of f has only sine terms. Moreover

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

where

$$b_n = \frac{2}{p} \int_0^p f(x) \sin\left(\frac{n\pi x}{p}\right)\, dx.$$

Half Range Sine and Half Range Cosine Series

Suppose f is only defined for $0 < x < p$. We can **extend** f to the left, to the interval $(-p, 0)$, as either an even function or as an odd function. Then we can express f with **two distinct** series:

Half range cosine series $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

where $$a_0 = \frac{2}{p}\int_0^p f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$$

Half range sine series $$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

where $$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$$

Find the Half Range Sine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right) \qquad p = 2$$

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$u = 2 - x \qquad du = -dx$$

$$= \int_0^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$dv = \sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{-2}{n\pi}(2-x)\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2 - \frac{2}{n\pi}\int_0^2 \cos\left(\frac{n\pi x}{2}\right)dx \qquad v = \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$$

0

$$= \frac{-2}{n\pi}\left((2-2)\cos\left(\frac{n\pi 2}{2}\right) - (2-0)\cos(0)\right)$$

$$= \frac{-2}{n\pi}(-2) = \frac{4}{n\pi}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi x}{2}\right)}$$

Find the Half Range Cosine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx \qquad a_n = \frac{2}{2}\int_0^L f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$a_0 = \int_0^2 (2-x)\,dx = 2x - \frac{x^2}{2}\Big|_0^2 = 4-2-0 = 2$$

$$u = 2-x \qquad du = -dx$$

$$a_n = \int_0^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx \qquad dv = \cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{2}{n\pi}(2-x)\sin\left(\frac{n\pi x}{2}\right)\Big|_0^2 + \frac{2}{n\pi}\int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx \qquad v = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$$

$$0$$

$$= -\left(\frac{2}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{2}\right)\Big|_0^2$$

$$= -\frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi 2}{2}\right) - \cos(0)\right]$$

$$= \frac{-4}{n^2\pi^2}\left((-1)^n - 1\right) = \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)$$

$$\boxed{f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)\cos\left(\frac{n\pi x}{2}\right)}$$

Plots of f with Half range series

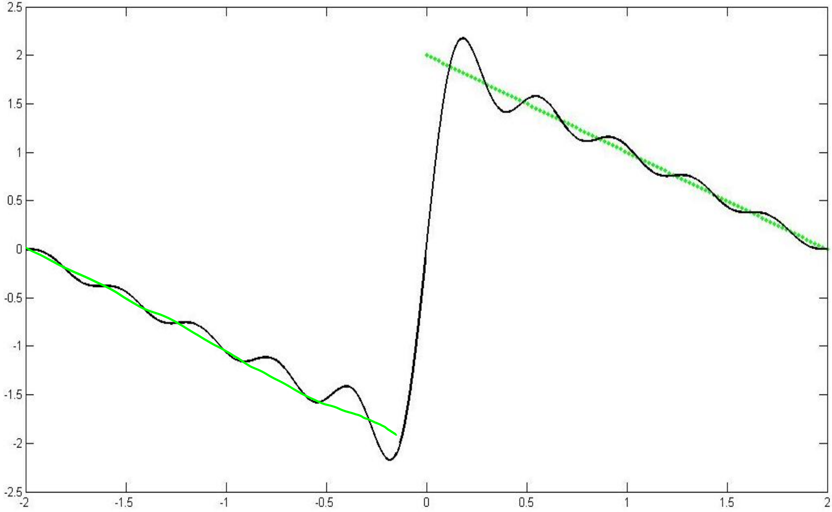


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series.

Plots of f with Half range series

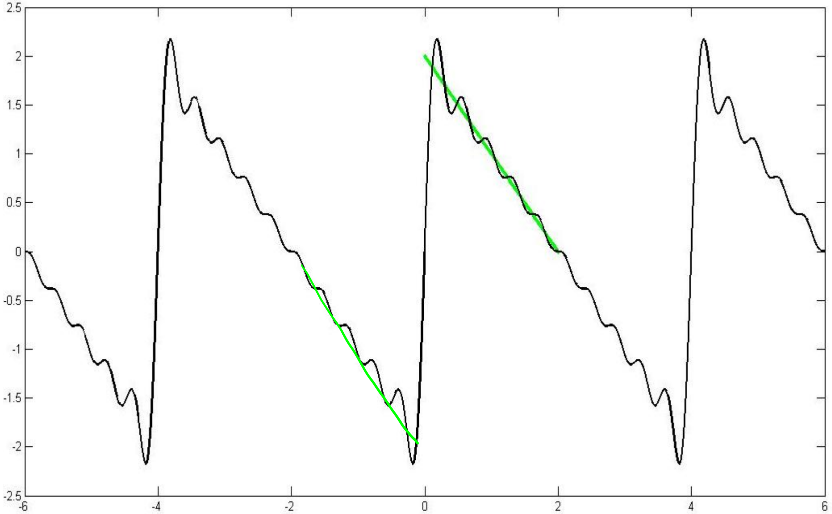


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series, and the series plotted over $(-6, 6)$

Plots of f with Half range series

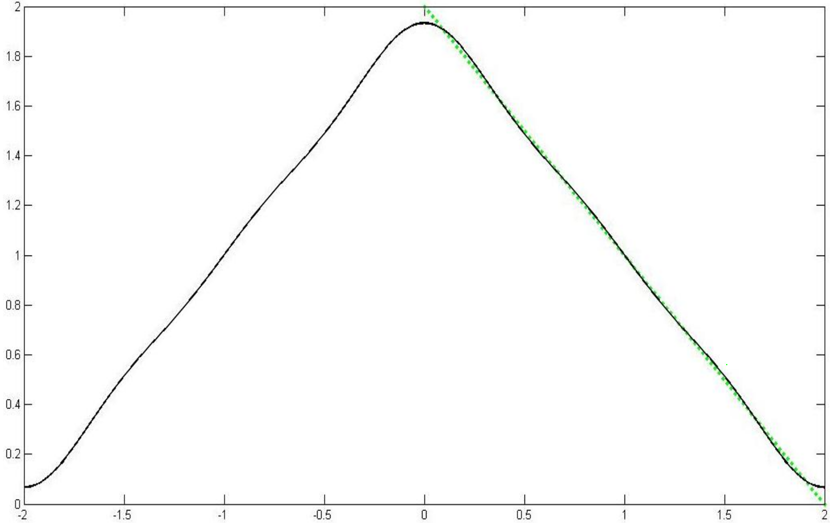


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series.

Plots of f with Half range series

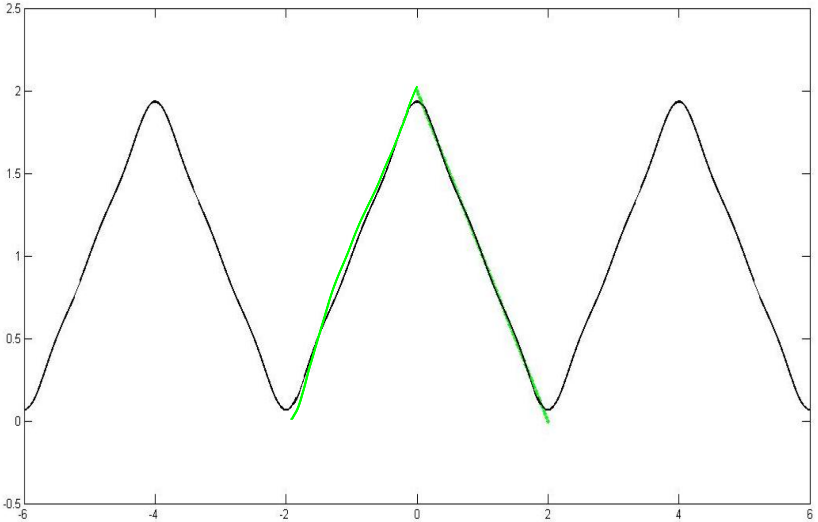
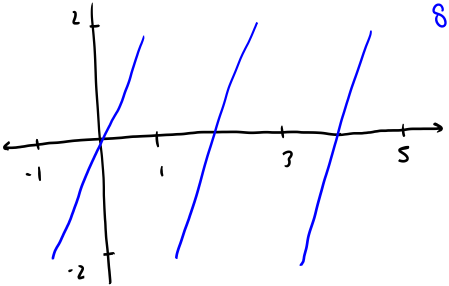


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series, and the series plotted over $(-6, 6)$

Solution of a Differential Equation

An undamped spring mass system has a mass of 2 kg attached to a spring with spring constant 128 N/m. The mass is driven by an external force $f(t) = 2t$ for $-1 < t < 1$ that is 2-periodic so that $f(t+2) = f(t)$ for all $t > 0$. Determine a particular solution x_p for the displacement for $t > 0$.



graph of f

The ODE is

$$mx'' + kx = f(t)$$

We can express the odd, periodic function f as a Fourier series

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t)$$

On Nov. 16, we found

$$f(t) = 2 \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

Our ODE is

$$2x'' + 128x = 2 \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

In standard form

$$x'' + 64x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

$$= \frac{2}{\pi}\sin(\pi t) - \frac{2}{2\pi}\sin(2\pi t) + \frac{2}{3\pi}\sin(3\pi t) - \dots$$

Using the method of undetermined coefficients set

$$x_p = \sum_{n=1}^{\infty} A_n \cos(n\pi t) + B_n \sin(n\pi t)$$

$$x_p' = \sum_{n=1}^{\infty} \left(-n\pi A_n \sin(n\pi t) + n\pi B_n \cos(n\pi t) \right)$$

$$x_p'' = \sum_{n=1}^{\infty}\left(-(n\pi)^2 A_n \cos(n\pi t) - (n\pi)^2 B_n \sin(n\pi t)\right)$$

$$x_p'' + 64x_p = \sum_{n=1}^{\infty} -(n\pi)^2 A_n \cos(n\pi t) - (n\pi)^2 B_n \sin(n\pi t)$$

$$+ \sum_{n=1}^{\infty} 64 A_n \cos(n\pi t) + 64 B_n \sin(n\pi t)$$

$$= \sum_{n=1}^{\infty}\left[\left(-(n\pi)^2 A_n + 64 A_n\right)\cos(n\pi t) + \left(-(n\pi)^2 B_n + 64 B_n\right)\sin(n\pi t)\right]$$

$$= \sum_{n=1}^{\infty}\left[0 \cdot \cos(n\pi t) + \frac{2(-1)^{n+1}}{n\pi}\sin(n\pi t)\right]$$

zero

$f(t)$

Matching coefficients

$$-(n\pi)^2 A_n + 64 A_n = 0$$

$$-(n\pi)^2 B_n + 64 B_n = \frac{2(-1)^{n+1}}{n\pi}$$

$$(64 - n^2\pi^2) A_n = 0 \qquad \Rightarrow \quad A_n = 0 \text{ for all } n$$

$$(64 - n^2\pi^2) B_n = \frac{2(-1)^{n+1}}{n\pi} \Rightarrow B_n = \frac{2(-1)^{n+1}}{n\pi(64 - n^2\pi^2)}$$

$64 - n^2\pi^2 \neq 0$ for all n

So

$$x_p = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi(64-n^2\pi^2)} \sin(n\pi t)$$