November 5 Math 2306 sec. 53 Fall 2018

Section 15: Shift Theorems

Theorem: (translation in s) Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \quad\Longrightarrow\quad \mathscr{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{s^2+k^2} \quad\Longrightarrow\quad \mathscr{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}.$$

Theorem (translation in t)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

In particular,

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{e^{-as}}{s}.$$

As another example,

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \implies \quad \mathscr{L}\{(t-a)^n\mathscr{U}(t-a)\} = \frac{n!e^{-as}}{s^{n+1}}.$$

A Couple of Useful Results

Another formulation of this translation theorem is

(1) $\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$

$g(t) = g((t+a)-a)$

Example: Find $\mathscr{L}\{\cos t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\} = e^{-\frac{\pi}{2}s}\mathscr{L}\{\cos(t+\frac{\pi}{2})\}$

$a = \frac{\pi}{2}$

Note

$\cos(t+\frac{\pi}{2}) = \cos t\cos\pi/2 - \sin t\sin\pi/2$

$= \cos t\cdot 0 - \sin t\cdot 1$

$= -\sin t$

So $\mathcal{L}\{\cos t\, \mathcal{U}(t-\pi/2)\} = e^{-\pi/2\, s}\, \mathcal{L}\{\cos(t+\pi/2)\}$

$$= e^{-\pi/2\, s}\, \mathcal{L}\{-\sin t\}$$

$$= -e^{-\frac{\pi}{2}s}\left(\frac{1}{s^2+1^2}\right)$$

$$= \frac{-e^{-\frac{\pi}{2}s}}{s^2+1}$$

A Couple of Useful Results

The inverse form of this translation theorem is

$$(2) \quad \mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

Example: Find $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+1)}\right\}$

We need to find $f(t) = \mathscr{L}^{-1}\left\{\dfrac{1}{s(s+1)}\right\}$

We will do partial fraction decomp

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$1 = A(s+1) + Bs$$

$$\left.\begin{array}{ll} s=0 & 1=A \\ s=-1 & 1=-B \end{array}\right\} \Rightarrow \quad A=1 \quad \text{and} \quad B=-1$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-(-1)}\right\}$$

$$= 1 - e^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\} = f(t-2)\,\mathcal{U}(t-2)$$

$$= \left(1-e^{-(t-2)}\right)\mathcal{U}(t-2)$$

Section 16: Laplace Transforms of Derivatives and IVPs

Suppose f has a Laplace transform and that f is differentiable on $[0, \infty)$. Obtain an expression for the Laplace tranform of $f'(t)$. (Assume f is of exponential order c for some c.)

$$\mathscr{L}\left\{f'(t)\right\} = \int_0^\infty e^{-st} f'(t)\, dt$$

Integrate by parts

$u = e^{-st}$ $\quad du = -se^{-st}\,dt$

$v = f(t)$ $\quad dv = f'(t)\,dt$

$$= f(t)e^{-st}\Big|_0^\infty - \left(\int_0^\infty -se^{-st} f(t)\,dt\right)$$

$f(t)\,e^{-st} \to 0$ as $t \to \infty$ for sufficiently large s

$$= 0 - f(0)e^0 + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\,\mathcal{L}\{f(t)\}$$

If we call $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Transforms of Derivatives

If $\mathscr{L}\{f(t)\} = F(s)$, we have $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$. We can use this relationship recursively to obtain Laplace transforms for higher derivatives of f.

For example

$$\mathscr{L}\{f''(t)\} = s\mathscr{L}\{f'(t)\} - f'(0)$$

$$= s\left(sF(s) - f(0)\right) - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

Transforms of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\left\{y(t)\right\} = Y(s),$$

then

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0),$$

$$\vdots$$

$$\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).$$

Differential Equation

For constants a, b, and c, take the Laplace transform of both sides of the equation

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Let $\mathcal{L}\{y(t)\} = Y(s)$ and $\mathcal{L}\{g(t)\} = G(s)$

Take transform of both sides

$$\mathcal{L}\{ay'' + by' + cy\} = \mathcal{L}\{g\}$$

$$a\mathcal{L}\{y''\} + b\mathcal{L}\{y'\} + c\mathcal{L}\{y\} = G(s)$$

$$a\left(s^2Y(s) - sy(0) - y'(0)\right) + b\left(sY(s) - y(0)\right) + cY(s) = G(s)$$

$Y_1(s)$ is the Laplace transform of the solution to the IVP with $y(0) = y_0$ and $y'(0) = 1$.

Let's isolate $Y(s)$.

$$as^2 Y(s) - as\,y(0) - ay'(0) + bs\,Y(s) - by(0) + c\,Y(s) = G(s)$$

$$as^2 Y(s) - as\,y_0 - ay_1 + bs\,Y(s) - by_0 + c\,Y(s) = G(s)$$

$$as^2 Y(s) + bs\,Y(s) + c\,Y(s) = ay_0 s + ay_1 + by_0 + G(s)$$

$$(as^2 + bs + c)\,Y(s) = ay_0 s + ay_1 + by_0 + G(s)$$

$$Y(s) = \frac{ay_0 s + ay_1 + by_0}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}$$

Note that $as^2 + bs + c$ is the characteristic polynomial for the original ODE

And the solution to the IVP

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

Solving IVPs

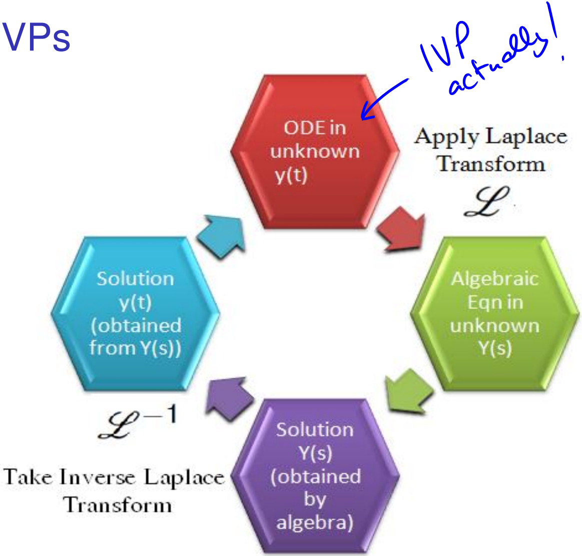

Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

General Form

We get

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where Q is a polynomial with coefficients determined by the initial conditions, G is the Laplace transform of $g(t)$ and P is the **characteristic polynomial** of the original equation.

$$\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}$$ is called the **zero input response**,

and

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$$ is called the **zero state response.**