Nov. 7 Math 1190 sec. 51 Fall 2016

Section 4.8: Antiderivatives; Differential Equations

Definition: A function F is called an antiderivative of f on an interval I if

$$F'(x) = f(x) \quad \text{for all } x \text{ in } I.$$

For example, $F(x) = x^2$ is an antiderivative of $f(x) = 2x$ on $(-\infty, \infty)$. Similarly, $G(x) = \tan x + 7$ is an antiderivative of $g(x) = \sec^2 x$ on $(-\pi/2, \pi/2)$.

Theorem: If F is any antiderivative of f on an interval I, then the *most general* antiderivative of f on I is

$$F(x) + C \quad \text{where } C \text{ is an arbitrary constant.}$$

Find the most general antiderivative of f.

(i) $f(x) = \cos x \quad I = (-\infty, \infty)$

• Find one example of an antiderivative, then add C.

$F(x) = \sin x$ since $\frac{d}{dx} \sin x = \cos x$

$\boxed{\sin x + C}$

(ii) $f(x) = \frac{1}{x} \quad I = (0, \infty)$

Note $\frac{d}{dx} \ln x = \frac{1}{x}$

$\boxed{\ln x + C}$

Question: Find the most general antiderivative of f.

(iii) $f(x) = \dfrac{1}{1+x^2}$ $\quad I = (-\infty, \infty)$

$$\frac{d}{dx} \ln(1+x^2) = \frac{1}{1+x^2} \cdot 2x = \frac{2x}{1+x^2}$$

(a) $F(x) = \dfrac{x}{x + x^3/3} + C$

(b) $F(x) = \ln(1+x^2) + C$

(c) $F(x) = \tan^{-1} x + C$

$$\frac{d}{dx}\left(\tan^{-1} x + c\right) = \frac{1}{1+x^2} + 0 = \frac{1}{1+x^2}$$

(iv) $f(x) = \sec x \tan x$ $\quad I = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

Since $\frac{d}{dx} \sec x = \sec x \tan x$

(a) $F(x) = \sec^2 x + C$

(b) $F(x) = \sec x + C$

(c) $F(x) = \tan x + C$

Find the most general antiderivative of

$$f(x) = x^n, \quad \text{where } n = 1, 2, 3, \ldots$$

This is a power function, so we'll think about the power rule.

To end up with power n, we must start with power $n+1$.

Let's guess that $F(x) = Ax^{n+1}$ for constant A.

We need $F'(x) = x^n$. We have

$$F'(x) = A(n+1)x^{n+1-1} = A(n+1)x^n$$

These match if $A(n+1) = 1 \Rightarrow A = \frac{1}{n+1}$.

So $F(x) = \frac{1}{n+1}x^{n+1} = \frac{x^{n+1}}{n+1}$

The most general antiderivative is

$$\frac{x^{n+1}}{n+1} + C.$$

power rule for antiderivatives

Some general results*:

(See the table on page 330 in Sullivan & Miranda for a more comprehensive list.)

Function	Particular Antiderivative	Function	Particular Antiderivative
$cf(x)$	$cF(x)$	$\cos x$	$\sin x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sin x$	$-\cos x$
x^n, $n \neq -1$	$\frac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
$\frac{1}{x}$	$\ln \lvert x \rvert$	$\csc x \cot x$	$-\csc x$
$\frac{1}{x^2+1}$	$\tan^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$

*We'll use the term particular antiderivative to refer to any antiderivative that has no arbitrary constant in it.

Example

Find the most general antiderivative of $h(x) = x\sqrt{x}$ on $(0, \infty)$.

$$h(x) = x\sqrt{x} = x \cdot x^{1/2} = x^{3/2}$$

Power rule

$$x^n \longrightarrow \frac{x^{n+1}}{n+1}$$

for $n \neq -1$

$$H(x) = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C$$

$$H(x) = \frac{2}{5} x^{5/2} + C$$

check $H'(x) = \frac{2}{5}\left(\frac{5}{2} x^{5/2 - 1}\right) + 0 = \frac{2}{5} \cdot \frac{5}{2} x^{3/2} = x^{3/2} = x\sqrt{x}$ ✓

Example

Determine the function $H(x)$ that satisfies the following conditions

$$H'(x) = x\sqrt{x}, \quad \text{for all } x > 0, \text{ and } H(1) = 0.$$

From the previous example, we know that

$$H(x) = \frac{2}{5} x^{5/2} + C .$$

$$H(1) = \frac{2}{5}(1)^{5/2} + C = 0 \Rightarrow \frac{2}{5} + C = 0 \Rightarrow C = \frac{-2}{5}$$

The solution to the whole problem, both conditions, is

$$H(x) = \frac{2}{5} x^{5/2} - \frac{2}{5} .$$

Example

A particle moves along the x-axis so that its acceleration at time t is given by

$$a(t) = 12t - 2 \quad \text{m/sec}^2.$$

At time $t = 0$, the velocity v and position s of the particle are known to be

$$v(0) = 3 \quad \text{m/sec, and} \quad s(0) = 4 \text{ m.}$$

Find the position $s(t)$ of the particle for all $t > 0$.

$a(t) = v'(t)$ to find v, we need an antiderivative of a.

$$v(t) = 12\frac{t^{1+1}}{1+1} - 2t + C = 6t^2 - 2t + C$$

Double check $v'(t) = 6(2t) - 2 + 0 = 12t - 2$

$v(t) = 6t^2 - 2t + C$ and $v(0) = 3$ m/sec.

$v(0) = 6\cdot 0^2 - 2\cdot 0 + C = 3 \Rightarrow C = 3$

The velocity $v(t) = 6t^2 - 2t + 3$

Note that $v(t) = s'(t)$ for position s.

we take an antiderivative to find s.

$$s(t) = 6\,\frac{t^{2+1}}{2+1} - 2\,\frac{t^{1+1}}{1+1} + 3t + C$$

$$s(t) = 2t^3 - t^2 + 3t + C$$

Double check

$$s'(t) = 2(3t^2) - 2t + 3 + 0 = 6t^2 - 2t + 3$$

Since $s(0) = 4$ was given

$$s(0) = 2 \cdot 0^3 - 0^2 + 3 \cdot 0 + C = 4 \Rightarrow C = 4$$

Our position @ time t is

$$\boxed{s(t) = 2t^3 - t^2 + 3t + 4}$$

Example

A **differential equation** is an equation that involves the derivative(s) of an unknown function. **Solving** such an equation would mean finding such an unknown function. Additional conditions are typically included in problems involving differential equations.

Solve the differential equation subject to the given *initial* conditions.

$$\frac{d^2y}{dx^2} = \cos x + 2, \quad y(0) = 0, \quad y'(0) = -1$$

We can get $y'(x)$ by taking an antiderivative.

$$y'(x) = \sin x + 2x + C$$

$y'(0) = -1 \Rightarrow y'(0) = \sin(0) + 2\cdot 0 + C = -1$

$C = -1$

$y'(x) = \sin x + 2x - 1$

Take another antiderivative.

$y(x) = -\cos x + 2\frac{x^{1+1}}{1+1} - x + C$

$= -\cos x + x^2 - x + C$

Check

$y'(x) = -(-\sin x) + 2x - 1 + 0 = \sin x + 2x - 1$

$y(x) = -\cos x + x^2 - x + C \quad , \quad y(0) = 0$

$y(0) = -\cos 0 + 0^2 - 0 + C = 0$

$-1 + C = 0 \Rightarrow C = 1$

The solution is

$$\boxed{y(x) = -\cos x + x^2 - x + 1}$$

Question

Solve the differential equation subject to the initial condition.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \quad y(0) = -\frac{1}{2}$$

(a) $y = \sin x - \frac{1}{2}$

(b) $y = \sqrt{1-x^2} - \frac{3}{2}$

(c) $y = \sin^{-1} x - \frac{1}{2}$

$$\frac{d}{dx}\left(\sin^{-1} x - \frac{1}{2}\right) = \frac{1}{\sqrt{1-x^2}} - 0$$

$$= \frac{1}{\sqrt{1-x^2}}$$

Section 5.1: Area (under the graph of a nonnegative function)

We will investigate the area enclosed by the graph of a function f. We'll make the following assumptions (for now):

- f is continuous on the interval $[a, b]$, and
- f is nonnegative, i.e $f(x) \geq 0$, on $[a, b]$.

Our Goal: Find the area of such a region.

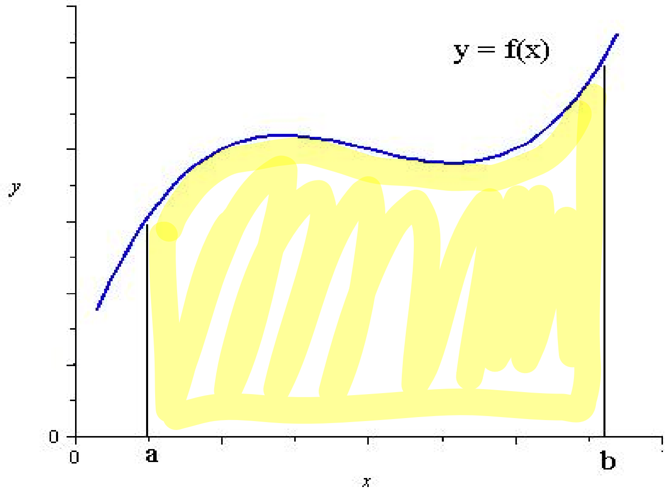

Figure: Region under a positive curve $y = f(x)$ on an interval $[a, b]$.

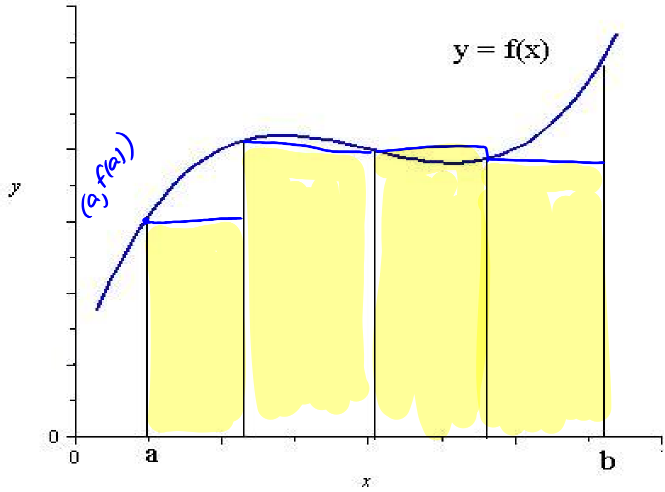

Figure: We could approximate the area by filling the space with rectangles.

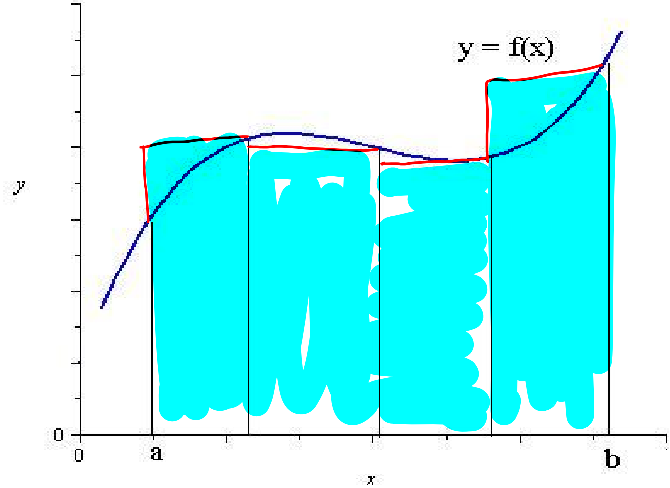

Figure: We could approximate the area by filling the space with rectangles.

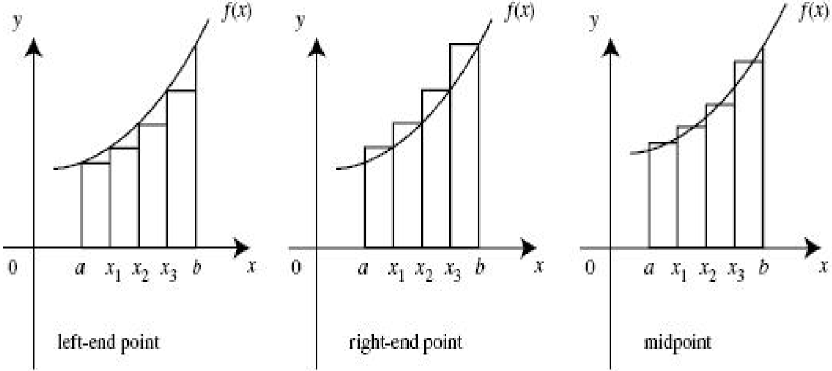

Figure: Some choices as to how to define the heights.

Approximating Area Using Rectangles

We can experiment with

- Which points to use for the heights (left, right, middle, other....)
- How many rectangles we use

to try to get a good approximation.

Definition: We will define the true area to be value we obtain taking the limit as the number of rectangles goes to $+\infty$.