November 7 Math 2306 sec. 56 Fall 2017

Section 16: Laplace Transforms of Derivatives and IVPs

A 1 kg mass is attached to a spring with spring constant 10 N/m. A dashpot induces damping numerically equation to 2 times the instantaneous velocity. A driving force of $f(t) = 18e^{-t}$ is imposed. If the mass starts at equilibruim from rest, determine the displacement for $t > 0$. Use the method of Laplace transforms.

$mx'' + \beta x' + kx = f(t) \qquad m=1,\ \beta=2,\ k=10$

$x'' + 2x' + 10x = 18e^{-t} \qquad x(0)=0,\ x'(0)=0$

let $\mathcal{L}\{x(t)\} = X(s)$

$\mathcal{L}\{x'' + 2x' + 10x\} = \mathcal{L}\{18e^{-t}\}$

$$\mathcal{L}\{x''\} + 2\mathcal{L}\{x'\} + 10\mathcal{L}\{x\} = 18\,\mathcal{L}\{e^{-t}\}$$

$$s^2 X(s) - s\underbrace{x(0)}_{0} - \underbrace{x'(0)}_{0} + 2\left(sX(s) - \underbrace{x(0)}_{0}\right) + 10X(s) = \frac{18}{s+1}$$

$$(s^2+2s+10)\,X(s) = \frac{18}{s+1}$$

$$X(s) = \frac{18}{(s+1)(s^2+2s+10)}$$

$s^2+2s+10$

Discriminant

$b^2-4ac = 2^2 - 4\cdot 1\cdot 10$

$= 4-40 < 0$

It's irreducible

Partial fractions

$$\frac{18}{(s+1)(s^2+2s+10)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+10}$$

$$18 = A(s^2+2s+10) + (Bs+C)(s+1)$$

$$= A(s^2+2s+10) + B(s^2+s) + C(s+1)$$

$$0s^2 + 0s + 18 = (A+B)s^2 + (2A+B+C)s + 10A+C$$

$$A+B=0 \Rightarrow B=-A$$

$$2A+B+C=0 \Rightarrow 2A-A+C=0$$

$$10A+C=18$$

$$\begin{array}{r} A+C=0 \\ 10A+C=18 \\ \hline -9A=-18 \end{array} \Rightarrow A=2$$

$B = -A = -2, \qquad C = -A = -2$

$$X(s) = \frac{2}{s+1} - \frac{2s+2}{s^2+2s+10}$$

Complete the square

$$s^2+2s+10 = s^2+2s+1+9$$

$$= (s+1)^2+9$$

$$X(s) = \frac{2}{s+1} - 2\,\frac{s+1}{(s+1)^2+3^2}$$

$$\frac{s+1}{(s+1)^2+3^2} \text{ looks like } \frac{s}{s^2+3^2} = \mathcal{L}\{\cos(3t)\}$$

shifted

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\}$$

$$\boxed{x(t) = 2e^{-t} - 2e^{-t}\cos(3t)}$$

$$* \quad \mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}$$

so

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\}$$