November 7 Math 2306 sec. 57 Fall 2017

Section 16: Laplace Transforms of Derivatives and IVPs

A 1 kg mass is attached to a spring with spring constant 10 N/m. A dashpot induces damping numerically equation to 2 times the instantaneous velocity. A driving force of $f(t) = 18e^{-t}$ is imposed. If the mass starts at equilibruim from rest, determine the displacement for $t > 0$. Use the method of Laplace transforms.

$mx'' + \beta x' + kx = f(t)$ $\quad m = 1,\ k = 10,\ \beta = 2$

$x'' + 2x' + 10x = 18e^{-t}$ $\quad x(0) = 0,\ x'(0) = 0$

$\mathcal{L}\{x'' + 2x' + 10x\} = \mathcal{L}\{18e^{-t}\}$ $\quad$ Let $\mathcal{L}\{x(t)\} = X(s)$

$$\mathcal{L}\{x''\} + 2\mathcal{L}\{x'\} + 10\mathcal{L}\{x\} = 18\mathcal{L}\{e^{-t}\}$$

$$s^2X(s) - sx(0) - x'(0) + 2\left(sX(s) - x(0)\right) + 10X(s) = \frac{18}{s+1}$$

($x(0) = 0$, $x'(0) = 0$, $x(0) = 0$)

$$(s^2 + 2s + 10)X(s) = \frac{18}{s+1}$$

$$X(s) = \frac{18}{(s+1)(s^2+2s+10)}$$

$s^2+2s+10$ discriminant $b^2-4ac = 2^2 - 4\cdot1\cdot10 = 4-40 < 0$

16' irreducible.

Partial fractions $\dfrac{18}{(s+1)(s^2+2s+10)} = \dfrac{A}{s+1} + \dfrac{Bs+C}{s^2+2s+10}$

$$18 = A(s^2+2s+10) + (Bs+C)(s+1)$$

$$18 = A(s^2+2s+10) + B(s^2+s) + C(s+1)$$

$$0s^2 + 0s + 18 = (A+B)s^2 + (2A+B+C)s + 10A + C$$

$$\left.\begin{aligned} A+B &= 0 \\ 2A+B+C &= 0 \\ 10A+C &= 18 \end{aligned}\right\}$$

$$A+B=0 \Rightarrow B=-A$$

$$\Rightarrow 2A - A + C = 0$$

$$A + C = 0$$

$$10A + C = 18$$
$$A + C = 0$$

$$9A = 18 \Rightarrow A = 2$$

$$B = -A = -2 \quad \text{and} \quad C = -A = -2$$

$$X(s) = \frac{2}{s+1} - \frac{2s+2}{s^2+2s+10}$$

Complete the square

$$s^2+2s+10 = s^2+2s+1+9 = (s+1)^2+3^2$$

$$X(s) = \frac{2}{s+1} - 2\,\frac{s+1}{(s+1)^2+3^2}$$

$\dfrac{s+1}{(s+1)^2+3^2}$ is $\dfrac{s}{s^2+3^2}$ with s shifted by -1.

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\}$$

$$\boxed{x(t) = 2e^{-t} - 2e^{-t}\cos(3t)}$$