November 7 Math 3260 sec. 57 Fall 2017

Section 5.1: Eigenvectors and Eigenvalues

Definition: Let A be an $n \times n$ matrix. A nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ is called an **eigenvector** of the matrix A.

A scalar λ such that there exists a nonzero vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is called an **eigenvalue** of the matrix A. Such a nonzero vector $\mathbf{x}$ is an *eigenvector corresponding to* λ.

Eigenspace

Definition: Let A be an $n \times n$ matrix and λ and eigenvalue of A. The set of all eigenvectors corresponding to λ—i.e. the set

$$\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \neq \mathbf{0} \text{ and } A\mathbf{x} = \lambda\mathbf{x}\},$$

is called the **eigenspace of A corresponding to λ.**

Remark: When combined with the zero vector, the eigenspace is the same as the null space of the matrix $A - \lambda I$. It follows that the eigenspace (along with $\mathbf{0}$) is a subspace of $\mathbb{R}^n$.

Matrices with Nice Structure

Theorem: If A is an $n \times n$ triangular matrix, then the eigenvalues of A are its diagonal elements.

Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

The eigenvalues are

$\lambda_1 = 3, \quad \lambda_2 = 2, \quad \lambda_3 = 1$

Example

Find at least one eigenvector for each eigenvalue of the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

For $\lambda_1 = 3$, $A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -1 & 0 \\ -1 & 0 & -2 \end{bmatrix}$

To solve $(A-3I)\vec{x} = \vec{0}$

$$\text{rref}(A-3I) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -2x_3$

$x_2 = 4x_3$

x_3 - free

$$\vec{x}_1 = x_3 \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 2$

$$A - 2I = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

$$\text{rref}(A-2I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$

$x_3 = 0$

x_2 - free

$$\vec{x}_2 = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda_3 = 1$, $A - 1I = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$\text{rref}(A - I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = 0$

$x_2 = 0$

x_3 - free

$\vec{x}_3 = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

A set of eigenvectors is $\left\{ \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Example Continued....

Demonstrate that the eigenvectors found are linearly independent.

One approach is to show that $\det(B) \neq 0$

where $B = [\vec{x}_1 \ \vec{x}_2 \ \vec{x}_3]$.

$$B = \begin{bmatrix} -2 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \det(B) = -2 \cdot 1 \cdot 1 = -2 \neq 0$$

Hence B has linearly independent.

Example

Suppose $\lambda = 0$ is an eigenvalue[1] of a matrix A. Argue that A is not invertible.

As an eigenvalue, there is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. If $\lambda = 0$, the equation is homogeneous $A\vec{x} = \vec{0}$.

Since there is a nontrivial solution to the homogeneous equation, A is singular - i.e. not invertible.

[1]Eigenvectors must be nonzero vectors, but it is perfectly legitimate to have a zero eigenvalue!

Theorems

Theorem: A square matrix A is invertible if and only if zero is **not** and eigenvalue.

Theorem: If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are eigenvectors of a matrix A corresponding to distinct eigenvalues, $\lambda_1, \dots, \lambda_p$, then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly independent.

Linear Independence

Show that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of a matrix A with corresponding eigenvalues λ_1 and λ_2 where $\lambda_1 \neq \lambda_2$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

Consider the equation $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$.

Suppose $\lambda_1 \neq 0$. Mult. by λ_1

$$c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_2 = \vec{0}. \qquad \text{eq } 1$$

Mult. by A

$$A(c_1\vec{v}_1 + c_2\vec{v}_2) = A\vec{0} = \vec{0}$$

$$c_1 A\vec{v}_1 + c_2 A\vec{v}_2 = \vec{0}$$

$$c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0} \qquad \text{eq } 2$$

subtract eqn 2 from eqn 1

$$
\begin{array}{rl}
 & c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_2 = \vec{0} \\
- & c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0} \\
\hline
 & (\lambda_1-\lambda_2)c_2\vec{v}_2 = \vec{0}
\end{array}
$$

$\vec{v}_2 \neq \vec{0}$ as an eigen vector

$\lambda_1 - \lambda_2 \neq 0$ since $\lambda_1 \neq \lambda_2$

So $c_2 = 0$.

The original equation becomes

$$c_1\vec{v}_1 + \vec{0} = \vec{0}$$

$$c_1\vec{v}_1 = \vec{0}$$

$\vec{v}_1 \neq \vec{0}$ as an eigenvector. Hence $c_1 = 0$.

Since $c_1 = c_2 = 0$ has to hold,

$\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

Section 5.2: The Characteristic Equation

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ by appealing to the fact that the equation $A\mathbf{x} = \lambda I_2\mathbf{x}$ can be restated as:

Find a nontrivial solution of the homogeneous equation

$$(A - \lambda I_2)\mathbf{x} = \mathbf{0}.$$

A nontrivial solution requires $A-\lambda I$ is singular. This means its determinant is zero.

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

$$0 = \det(A - \lambda I) = (2-\lambda)(-6-\lambda) - 3\cdot 3$$

$$= \lambda^2 + 4\lambda - 12 - 9$$

$$0 = \lambda^2 + 4\lambda - 21$$

$$0 = (\lambda + 7)(\lambda - 3)$$

The two solutions are $\lambda_1 = -7$ and $\lambda_2 = 3$.

Theorem (adding more to the invertible matrix theorem)

The $n \times n$ matrix A is invertible if and only if[2]

(s) The number 0 is not an eigenvalue of A.

(t) The determinant of A is nonzero.

[2] This is nothing new, we're just adding to the list.

Characteristic Equation

Definition: For $n \times n$ matrix A, the expression

$$\det(A - \lambda I)$$

is an n^{th} degree polynomial in λ. It is called the **characteristic polynomial** of A.

Definition: The equation

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of A.

Theorem: The scalar λ is an eigenvalue of the matrix A if and only if it is a root of the characteristic equation.

Example

Find the characteristic equation for the matrix and identify all of its eigenvalues.

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

.

$$A - \lambda I = \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

$$= (5-\lambda)^2(3-\lambda)(1-\lambda)$$

This is the characteristic polynomial.

The characteristic equation is

$$(5-\lambda)^2(3-\lambda)(1-\lambda)=0.$$

The eigen values are 5, 3, and 1.

Multiplicities

Definition: The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation. The **geometric multiplicity** is the dimension of its corresponding eigenspace.

Example Find the algebraic and geometric multiplicity of the eigenvalue $\lambda = 5$ of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

.

Char. eqn

$$\underbrace{(\lambda-5)^2}_{\text{5 is double root}}(\lambda-3)(\lambda-1) = 0$$

The algebraic multiplicity of 5 is 2.

$$A-5I = \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & -8 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_1 - free

$x_2 = x_3 = x_4 = 0$

eigen vectors have the form $\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

A basis for the eigen space is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

The geometric multiplicity of 5 is 1.