Section 5.1: Eigenvectors and Eigenvalues

Definition: Let $A$ be an $n \times n$ matrix. A nonzero vector $x$ such that

$$Ax = \lambda x$$

for some scalar $\lambda$ is called an eigenvector of the matrix $A$.

A scalar $\lambda$ such that there exists a nonzero vector $x$ satisfying $Ax = \lambda x$ is called an eigenvalue of the matrix $A$. Such a nonzero vector $x$ is an eigenvector corresponding to $\lambda$. 
**Eigenspace**

**Definition:** Let $A$ be an $n \times n$ matrix and $\lambda$ and eigenvalue of $A$. The set of all eigenvectors corresponding to $\lambda$—i.e. the set

$$\{x \in \mathbb{R}^n \mid x \neq 0 \text{ and } Ax = \lambda x\},$$

is called the **eigenspace of $A$ corresponding to $\lambda$**.

**Remark:** When combined with the zero vector, the eigenspace is the same as the null space of the matrix $A - \lambda I$. It follows that the eigenspace (along with 0) is a subspace of $\mathbb{R}^n$. 
Theorem: If $A$ is an $n \times n$ triangular matrix, then the eigenvalues of $A$ are its diagonal elements.

Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

The eigenvalues are

$\lambda_1 = 3, \quad \lambda_2 = 2, \quad \lambda_3 = 1$
Example

Find at least one eigenvector for each eigenvalue of the matrix

\[ A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}. \]

For \( \lambda_1 = 3 \), \( A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -1 & 0 \\ -1 & 0 & -2 \end{bmatrix} \)

To solve \( (A - 3I) \vec{x} = \vec{0} \)

\( \text{ref} \ (A - 3I) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \)

\( x_1 = -2x \)

\( x_2 = 4x_3 \)

\( x_3 - \text{free} \)
The eigen vectors look like: 
\[ \hat{x}_1 = x_3 \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \]

For \( \lambda = 2 \)

\[ A - 2I = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix} \]

\[ \text{det} (A - 2I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 = 0 \\ x_2 = 0 \\ x_3 \text{ - free} \end{bmatrix} \]

The eigen vectors are:
\[ \hat{x}_2 = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \]
For \( \lambda_3 = 1 \), \( A - I = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \)

\[
\text{ref} \ (A - I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\( x_1 = 0 \quad x_2 = 0 \quad x_3 - \text{free} \)

\[
\Rightarrow x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

A set of eigenvectors is \( \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \).
Demonstrate that the eigenvectors found are linearly independent.

With 3 vectors in \( \mathbb{R}^3 \), we can use the square matrix:

\[
B = \begin{bmatrix}
\bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
-2 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
\end{bmatrix}
\]

\[
\det(B) = -2 \neq 0
\]

\( B \) is nonsingular, its columns are linearly independent.
**Theorems**

**Theorem:** A square matrix $A$ is invertible if and only if zero is **not** an eigenvalue.

**Theorem:** If $v_1, \ldots, v_p$ are eigenvectors of a matrix $A$ corresponding to distinct eigenvalues, $\lambda_1, \ldots, \lambda_p$, then the set $\{v_1, \ldots, v_p\}$ is linearly independent.
Linear Independence

Show that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of a matrix $A$ with corresponding eigenvalues $\lambda_1$ and $\lambda_2$ where $\lambda_1 \neq \lambda_2$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

Consider $c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = \mathbf{0}$.

Form an equation by multiplying by $\lambda_1$.

\[ c_1 \lambda_1 \mathbf{v}_1 + c_2 \lambda_1 \mathbf{v}_2 = \mathbf{0} \quad \text{eqn 1} \]

Form an equation by multiplying by $A$.

\[ A (c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2) = A \mathbf{0} = \mathbf{0} \]

\[ c_1 A \mathbf{v}_1 + c_2 A \mathbf{v}_2 = \mathbf{0} \]
\[ c_1 \lambda_1 \tilde{V}_1 + c_2 \lambda_2 \tilde{V}_2 = 0 \quad \text{eqn 2} \]

Subtract eqn 2 from eqn 1
\[ c_1 \lambda_1 \tilde{V}_1 + c_2 \lambda_1 \tilde{U}_2 = 0 \]
\[ c_1 \lambda_1 \tilde{V}_1 + c_2 \lambda_2 \tilde{U}_2 = 0 \]
\[ (\lambda_1 - \lambda_2) c_2 \tilde{V}_2 = 0 \]

\[ \tilde{V}_2 \neq 0 \text{ since it's an eigen vector} \]
\[ \lambda_1 - \lambda_2 \neq 0 \text{ since } \lambda_1 \neq \lambda_2 \]
so \[ c_2 = 0 \].
The original equation becomes

\[ c_1 \mathbf{v}_1 = \mathbf{0} \]

Since \( \mathbf{v}_1 \neq \mathbf{0} \), it must be that \( c_1 = 0 \).

Since it must be that \( c_1 = c_2 = 0 \),

\( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are linearly independent.
Section 5.2: The Characteristic Equation

Find the eigenvalues of \( A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} \) by appealing to the fact that the equation \( Ax = \lambda I_2 x \) can be restated as:

Find a nontrivial solution of the homogeneous equation

\[
(A - \lambda I_2)x = 0.
\]

We require the matrix \( A - \lambda I \) be singular.

It follows that \( \text{det}(A - \lambda I) \) must be zero.

\[
A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2 - \lambda & 3 \\ 3 & -6 - \lambda \end{bmatrix}
\]
\[
\det (A - \lambda I) = (2 - \lambda)(-6 - \lambda) - 3.3
\]
\[
= \lambda^2 + 4\lambda - 12 - 9 = \lambda^2 + 4\lambda - 21
\]

We want \( \lambda \) to solve

\[
0 = \lambda^2 + 4\lambda - 21
\]

\[
= (\lambda + 7)(\lambda - 3)
\]

There are two values \( \lambda_1 = -7 \) and \( \lambda_2 = 3 \).
Theorem (adding more to the invertible matrix theorem)

The $n \times n$ matrix $A$ is invertible if and only if\(^1\)

(s) The number 0 is not an eigenvalue of $A$.

(t) The determinant of $A$ is nonzero.

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\(^1\)This is nothing new, we’re just adding to the list.
Characteristic Equation

**Definition:** For $n \times n$ matrix $A$, the expression

$$\det(A - \lambda I)$$

is an $n^{th}$ degree polynomial in $\lambda$. It is called the **characteristic polynomial** of $A$.

**Definition:** The equation

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of $A$.

**Theorem:** The scalar $\lambda$ is an eigenvalue of the matrix $A$ if and only if it is a root of the characteristic equation.
Example

Find the characteristic equation for the matrix and identify all of its eigenvalues.

\[
A = \begin{bmatrix}
5 & -2 & 6 & -1 \\
0 & 3 & -8 & 0 \\
0 & 0 & 5 & 4 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
A - \lambda I = \begin{bmatrix}
5-\lambda & -2 & 6 & -1 \\
0 & 3-\lambda & -8 & 0 \\
0 & 0 & 5-\lambda & 4 \\
0 & 0 & 0 & 1-\lambda
\end{bmatrix}
\]

The characteristic polynomial is

\[
det(A - \lambda I) = (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)
\]

\[
= (5-\lambda)^2(3-\lambda)(1-\lambda)
\]
The characteristic equation is

$$(5-\lambda)^2(3-\lambda)(1-\lambda) = 0$$

The eigen values are 5, 3, and 1.
**Multiplicities**

**Definition:** The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation. The **geometric multiplicity** is the dimension of its corresponding eigenspace.

**Example** Find the algebraic and geometric multiplicity of the eigenvalue $\lambda = 5$ of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The characteristic eqn. is

$$(\lambda - 5)^2(\lambda - 3)(\lambda - 1) = 0$$

5 is a double root so its algebraic multiplicity is 2.
\[ A - 5I = \begin{bmatrix} 0 & -2 & 0 & 1 \\ 0 & -2 & -8 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ x_1 \text{ -free} \quad x_2 = x_3 = x_4 = 0 \]

The eigen vectors are
\[
\hat{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\]

The geometric multiplicity is 1.