November 9 MATH 1113 sec. 52 Fall 2018

Section 6.5: Trigonometric Functions of a Real Variable and Their Graphs

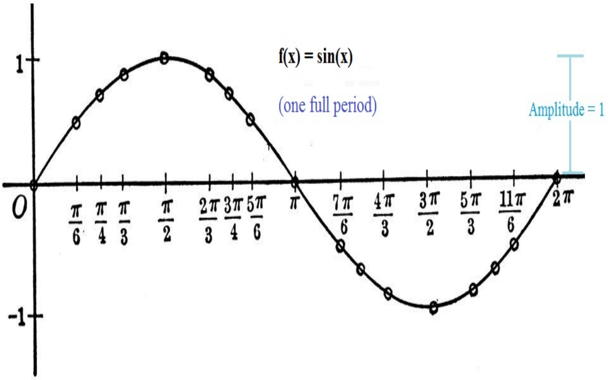


Figure: It's worth noting that major features (maximum, minimum, x-intercepts) divide the period into four equal parts.

Plot of $y = f(x) = \sin x$

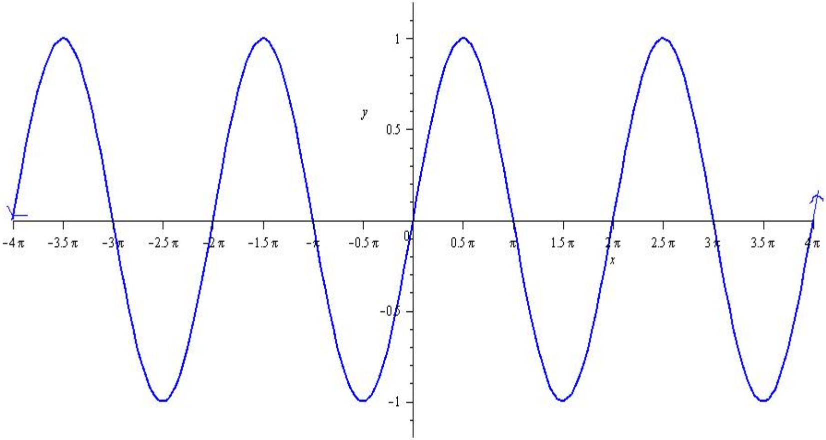


Figure: $y = \sin x$ over four full periods. Note the periodicity and the odd symmetry.

Plot of $y = f(x) = \cos x$

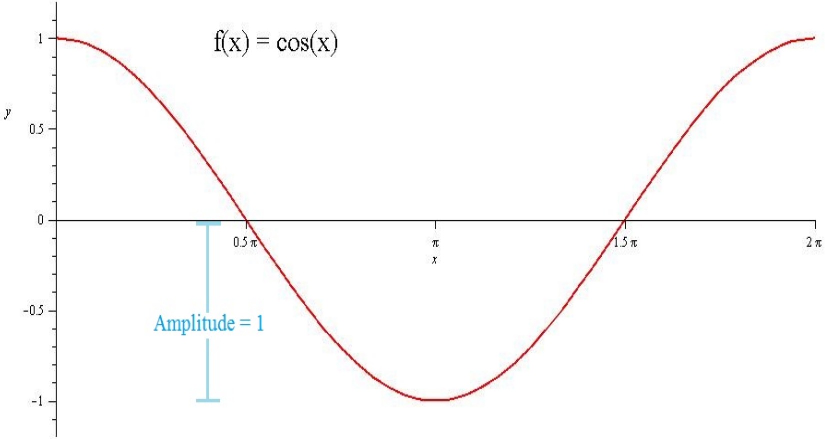


Figure: $y = \cos x$ over one full period.

Plot of $y = f(x) = \cos x$

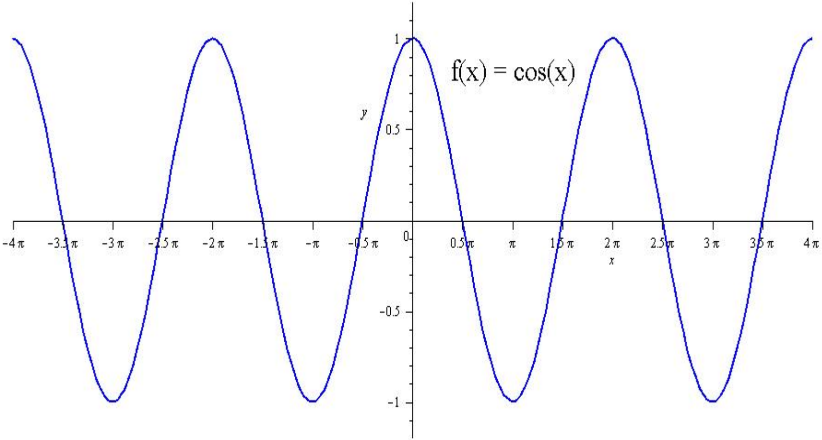


Figure: $y = \cos x$ over four full periods. Note the periodicity and the even symmetry

Plot of $f(x) = \cos x$ and $g(x) = \sin x$ Together

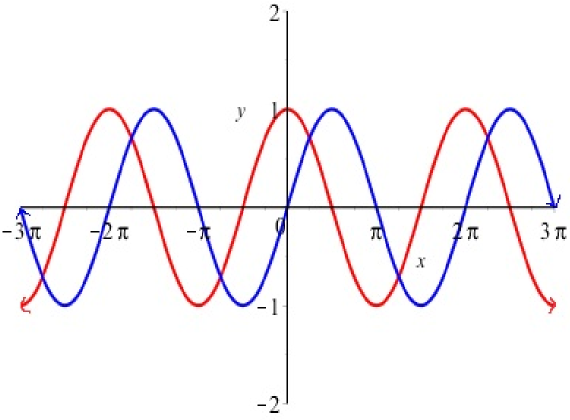


Figure: Note that the cofunction property $\cos(x) = \sin(\pi/2 - x)$ can be seen in the graphs.

A word of caution on using the unit circle

Our goal is to know and understand the properties of the trigonometric functions as **real valued functions of real variables**.

- They appear in many models of periodic processes (sound/water waves, seasonal processes, predator-prey interactions) that have no connection to triangles or circles.
- Right triangles and the unit circle provide powerful mnemonic devices.
- You'll often see x used as the independent variable—in this class and in Calculus to be sure! The point $(x, y) = (\cos s, \sin s)$ on the unit circle are not the same (x, y) as in $y = \cos x$. Get used to it!

Summary of Sine and Cosine Properties

Let $f(x) = \sin x$ and $g(x) = \cos x$. Then

- Both f and g are periodic with fundamental period 2π.
- The domain of both f and g is $(-\infty, \infty)$.
- The range of both f and g is $[-1, 1]$; they have amplitude 1.
- f is an odd function, i.e. $f(-x) = \sin(-x) = -\sin x = -f(x)$.
- g is an even function, i.e. $g(-x) = \cos(-x) = \cos x = g(x)$.
- Both f and g are continuous on their entire domain $(-\infty, \infty)$.
- The zeros of f are integer multiples of π, i.e. $f(n\pi) = 0$ for $n = 0, \pm 1, \pm 2, \ldots$
- The zeros of g are odd integer multiples of $\pi/2$, $g\left(\frac{m\pi}{2}\right) = 0$ for $m = \pm 1, \pm 3, \ldots$

Question

Let $f(x) = \sin x$, and suppose that $f(a) = -0.62$ for some real number a. Which of the following is true?

$\sin(x + 2\pi) = \sin(x)$ for all real x

(a) $f(a + 2\pi) = -0.62$ ✓

(b) $f(2a) = -1.24$ ← outside the range $-1 \le \sin(x) \le 1$ for all x

(c) $f(-a) = 0.62$ ✓

$\sin(-x) = -\sin(x)$ for all real x

(d) Only (a) and (b) are true.

(e) Only (a) and (c) are true.

Question

True or False: The equation $\cos x = 2$ has no real solutions.

(a) True, and I'm confident.

(b) True, but I'm not confident.

(c) False, and I'm confident.

(d) False, but I'm not confident.

True because $-1 \leq \cos x \leq 1$ for all real x

The Tangent

The function $\tan s = \dfrac{\sin s}{\cos s}$. Recall that

$$\cos s = 0 \quad \text{whenever} \quad s = \frac{m\pi}{2} \quad \text{for} \quad m = \pm 1, \pm 3, \pm 5, \ldots$$

When $\cos s = 0$, $\sin s$ is either 1 or -1. Hence

Domain: The domain of the tangent function is all real number **except** odd multiples of $\pi/2$. We can write this as

$$\left\{ s \;\middle|\; s \neq \frac{\pi}{2} + k\pi,\ k = 0, \pm 1, \pm 2, \ldots \right\}$$

Moreover, the graph of the tangent function has vertical asymptotes at each odd multiple of $\pi/2$.

The Tangent

Range: The range of the tangent function is **all real numbers.**

Symmetry: The function $f(s) = \tan s$ is odd. That is

$$f(-s) = \tan(-s) = -\tan s = -f(s).$$

Perodicity: The tangent function is periodic with fundamental period π. That is

$$\tan(s + \pi) = \tan s \quad \text{for all } s \text{ in the domain.}$$

Note: The period of the tangent function is π. This is different from the period of the sine and cosine.

The Tangent

A few key tangent values:

s	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan s$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef.

And due to symmetry

s	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0
$\tan s$	undef.	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

Here is an applet to plot two periods of the function $f(s) = \tan s$.

GeoGebra Graph Applet: Sine, Cosine, and Tangent

Basic Plot $f(x) = \tan x$

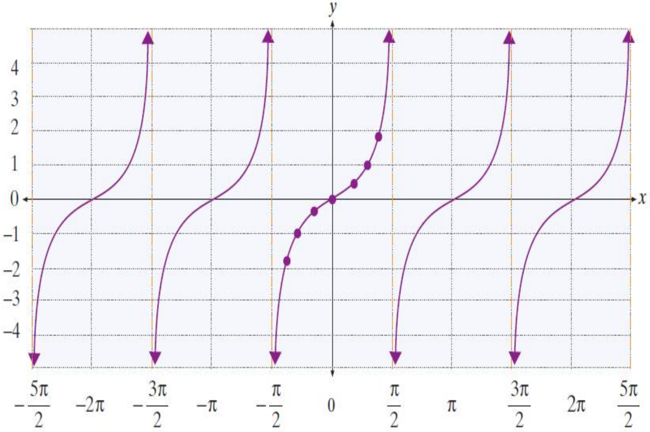


Figure: Plot of several periods of $f(x) = \tan x$. Note that the interval between adjacent asymptotes is the period π.

Cotangent: Using the Cofunction ID and Symmetry

$$\cot s = \tan\left(\frac{\pi}{2} - s\right) = \tan\left(-\left(s - \frac{\pi}{2}\right)\right) = -\tan\left(s - \frac{\pi}{2}\right).$$

So the graph of $f(s) = \cot s$ is the graph of $g(s) = \tan s$ under a horizontal shift $\pi/2$ units to the right followed by a reflection in the s-axis.

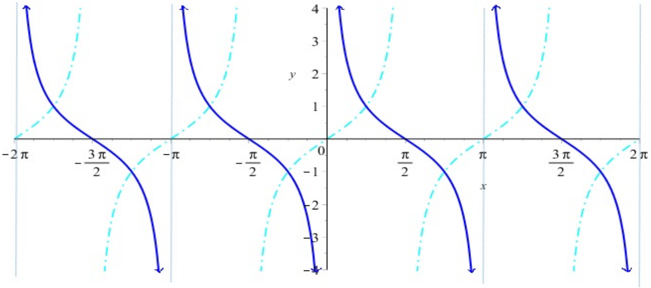


Figure: Plot of $f(x) = \cot x$. Note that the lines $x = n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$ are vertical asymptotes to the graph. The dashed curve is $y = \tan x$.

Cosecant and Secant

Domains: Since $\sin(n\pi) = 0$ for integers n,

$$\textbf{Domain}(\csc s) \;=\; \{s \,|\, s \neq n\pi, \text{ for integers } n\}.$$

Since $\cos\left(\frac{\pi}{2} + n\pi\right) = 0$ for integers n, the domain of $\sec s$ is

$$\textbf{Domain}(\sec s) \;=\; \left\{s \,\middle|\, s \neq \frac{\pi}{2} + n\pi, \text{ for integers } n\right\}.$$

Ranges: Note that

$$|\csc s| = \frac{1}{|\sin s|} \geq 1 \quad \text{and} \quad |\sec s| = \frac{1}{|\cos s|} \geq 1$$

so the range of both $\csc s$ and $\sec s$ is

$$(-\infty, -1] \cup [1, \infty).$$

Cosecant: Using $\csc s = \frac{1}{\sin s}$

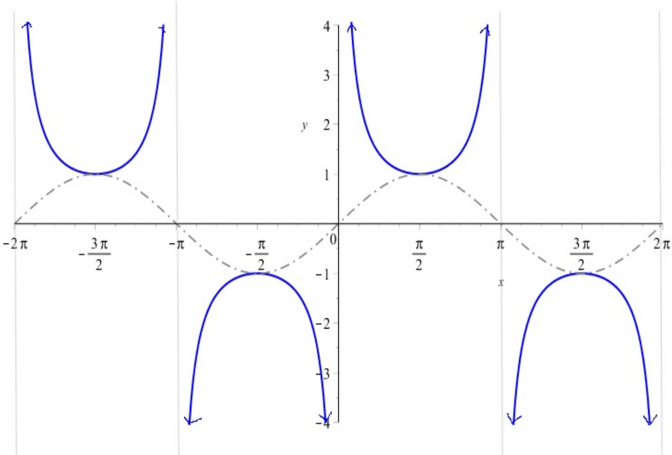


Figure: Two periods of $f(s) = \csc s$. The dashed curve is $y = \sin s$. Note the asymptotes $s = n\pi$ for integers n where $\sin s$ takes its zeros. The curves meet at the relative extrema and have the same period 2π.

Secant: Using $\sec s = \frac{1}{\cos s}$

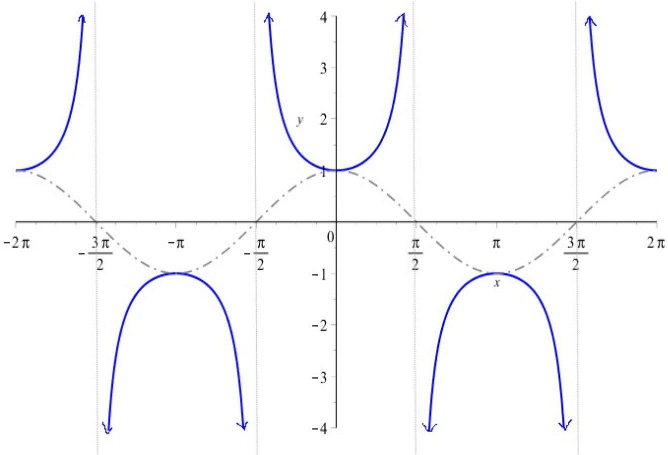


Figure: Two periods of $f(s) = \sec s$. The dashed curve is $y = \cos s$. Note the asymptotes $s = \pi/2 + n\pi$ for integers n where $\cos s$ takes its zeros. The curves meet at the relative extrema and have the same period 2π.

Question

Recall that the cosine is an even function and the sine is an odd function. Moreover,

$$\sec x = \frac{1}{\cos x} \quad \text{and} \quad \csc x = \frac{1}{\sin x}$$

Which of the following statements is true about the symmetry of the secant and cosecant.

(a) Both of the secant and cosecant are even functions.

(b) Both the secant and cosecant are odd functions.

(c) Secant is even, and cosecant is odd.

(d) Secant is odd, and cosecant is even.

(e) Symmetry for secant and cosecant can't be deduced from symmetry of cosine and sine.

Section 6.6: Graphing Trigonometric Functions with Transformations

Our goal is to graph functions of the form

$$f(x) = a\sin(bx - c) + d \quad \text{or} \quad f(x) = a\cos(bx - c) + d$$

Note: here we will be graphing points (x, y) on a curve $y = f(x)$.

Amplitude

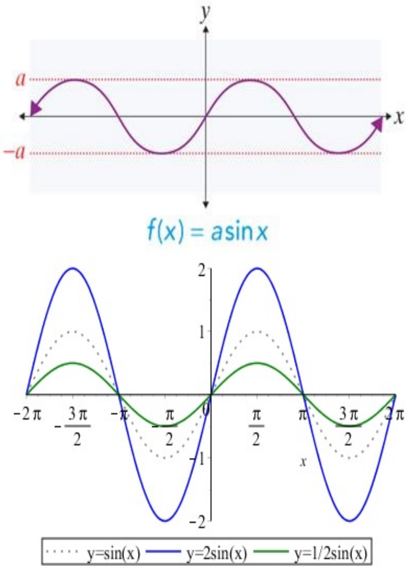
Consider: $f(x) = a\sin(bx - c) + d$ or $f(x) = a\cos(bx - c) + d$

Definition: Let a be any nonzero real number. The **amplitude** of the function f defined above is the value $|a|$.

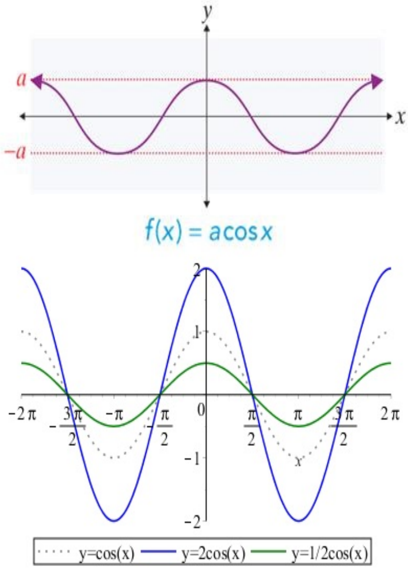
Recall that this is half the distance between the maximum and minimum values.

If $a < 0$ the graph is reflected in the x-axis. But the amplitude is still $|a|$.

Amplitude



y
a
x
−a
f(x) = asinx
2
1
−2
−2π
−3π/2
−π
0
π/2
π
3π/2
2π
x
y=sin(x)
y=2sin(x)
y=1/2sin(x)



y
a
x
−a
f(x) = acosx
2
1
−1
−2
−2π
−3π/2
−π
0
−π/2
π/2
π
3π/2
2π
x
y=cos(x)
y=2cos(x)
y=1/2cos(x)

Example

Identify the amplitude A of each function. Determine if the graph is reflected in the x-axis.

(a) $f(x) = 3\sin(4x-2)+1$ $A = 3$ no reflection as 3 is positive

(b) $f(x) = -5\sin\left(\frac{\pi x}{2}\right)+7$ $A = 5$ yes reflection as -5 is negative

(c) $f(x) = 2-6\cos(2x+3)$ $A = 6$ yes reflection -6 is negative

$= -6\cos(2x+3)+2$