Nov. 9 Math 1190 sec. 51 Fall 2016

Section 5.1: Area (under the graph of a nonnegative function)

We will investigate the area enclosed by the graph of a function f. We'll make the following assumptions (for now):

- f is continuous on the interval $[a, b]$, and
- f is nonnegative, i.e $f(x) \geq 0$, on $[a, b]$.

Our Goal: Find the area of such a region.

We'll start by approximating the region with a bunch of rectangles, then move to the exact value.

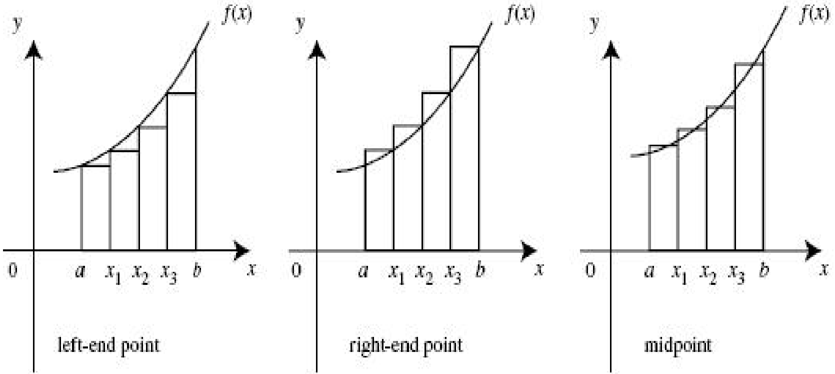


Figure: Some choices as to how to define the heights.

Approximating Area Using Rectangles

We can experiment with

- Which points to use for the heights (left, right, middle, other....)
- How many rectangles we use

to try to get a good approximation.

Definition: We will define the true area to be value we obtain taking the limit as the number of rectangles goes to $+\infty$.

Some terminology

- A **Partition** P of an interval $[a, b]$ is a collection of points $\{x_0, x_1, ..., x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

- A **Subinterval** is one of the intervals $x_{i-1} \leq x \leq x_i$ determined by a partition.
- The width of a subinterval is denoted $\Delta x_i = x_i - x_{i-1}$. If they are all the same size (equal spacing), then

$$\Delta x = \frac{b-a}{n}, \quad \text{and this is called the } \textbf{norm} \text{ of the partition.}$$

- A set of **sample points** is a set $\{c_1, c_2, \ldots, c_n\}$ such that $x_{i-1} \leq c_i \leq x_i$.

Taking the number of rectangles to ∞ is the same as taking the width $\Delta x \to 0$.

Example:

Write an equally spaced partition of the interval $[0, 2]$ with the specified number of subintervals, and determine the norm Δx.

(a) For $n = 4$

$\left\{ 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \right\}$

$x_0 = 0$

$x_1 = 1/2$

$x_2 = 1$

$x_3 = 3/2$

$x_4 = 2$

$a = 0$

$b = 2$

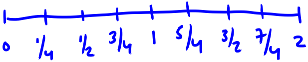
$n = 4$

$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$

Example:

Write an equally spaced partition of the interval $[0, 2]$ with the specified number of subintervals, and determine the norm Δx.

(b) For $n = 8$



$\left\{0, \frac{1}{4}, \frac{1}{2}, 3/4, 1, 5/4, 3/2, 7/4, 2\right\}$

$\Delta x = \frac{b-a}{n} = \frac{2-0}{8} = \frac{1}{4}$

$x_0 = 0$

$x_1 = 1/4 = 0 + 1 \cdot \frac{1}{4}$

$x_2 = 1/2 = 0 + 2 \cdot \frac{1}{4}$

$x_3 = 3/4 = 0 + 3 \cdot \frac{1}{4}$

$x_4 = 1$.

$x_5 = 5/4$.

$x_6 = 3/2$.

$x_7 = 7/4$

$x_8 = 2$

$x_i = 0 + i \frac{1}{4}$

Question

$\swarrow n=6$

Write an equally spaced partition of the interval $[0, 2]$ with 6 subintervals, and determine the norm Δx.

(a) $\left\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\right\} \quad \Delta x = \frac{1}{3}$

(b) $\left\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\right\} \quad \Delta x = \frac{1}{6}$

(c) $\left\{0, \frac{1}{6}, \frac{1}{3}, 1, \frac{5}{6}, \frac{7}{6}, 2\right\} \quad \Delta x = \frac{1}{3}$

(c) Find an equally spaced partition of $[0, 2]$ having N subintervals. What is the norm Δx?

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{N} = \frac{2}{N}$$

$x_N = 2$

$x_0 = 0$

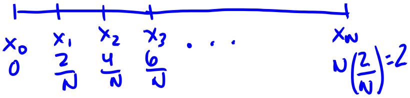
$x_1 = \frac{2}{N}$

$x_2 = 2 \cdot \frac{2}{N}$

$x_3 = 3 \cdot \frac{2}{N} \quad \Rightarrow \quad x_i = i\,\frac{2}{N} = \frac{2i}{N}$

Note $\quad x_N = N\left(\frac{2}{N}\right) = 2$

$$\left\{ x_i \;\middle|\; x_i = \frac{2i}{N},\ i = 0, \ldots, N \right\}$$



Approximating area with a Partition and sample points

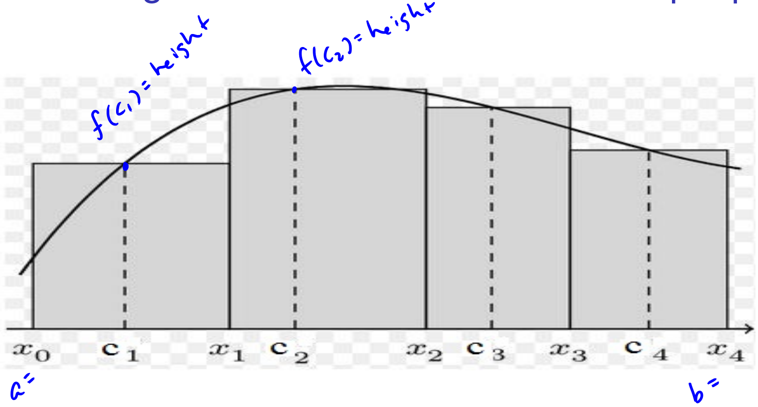


Figure: Area $\approx f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + f(c_4)\Delta x$. This can be written as

$$\sum_{i=1}^{n} f(c_i)\Delta x.$$

In general, an equally spaced partition of $[a, b]$ with n subintervals means

- $\Delta x = \frac{b-a}{n}$
- $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, i.e. $x_i = a + i\Delta x$
- Taking heights to be

$$\text{left ends} \quad c_i = x_{i-1} \quad \text{area} \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$\text{right ends} \quad c_i = x_i \quad \text{area} \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

- The true area exists (for f continuous) and is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

Lower and Upper Sums

The standard way to set up these sums is to take c_i such that

$$f(c_i) \quad \text{is the abs. minimum value of } f \text{ on } [x_{i-1}, x_i]$$

Then set A_L

$$A_L = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x.$$

This is called a **Lower Riemann sum**.

Lower and Upper Sums

Then, we take C_i such that

$$f(C_i) \quad \text{is the abs. maximum value of } f \text{ on } [x_{i-1}, x_i]$$

Then set A_U

$$A_U = \lim_{n\to\infty} \sum_{i=1}^{n} f(C_i)\Delta x.$$

This is called a **Upper Riemann sum**.

Lower and Upper Sums

If f is continuous on $[a, b]$, then it will necessarily be that

$$A_L = A_U.$$

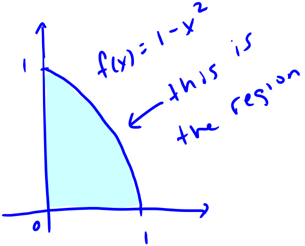
This value is the true area.

In practice, these are tough to compute unless f is only increasing or only decreasing. So instead, we tend to use left and right sums.

Example: Find the area under the curve $y = 1 - x^2$, $0 \leq x \leq 1$.

Use right end points $c_i = x_i$ and assume the following identity

$$\sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6} \quad \text{(sum of first } n \text{ squares)}$$

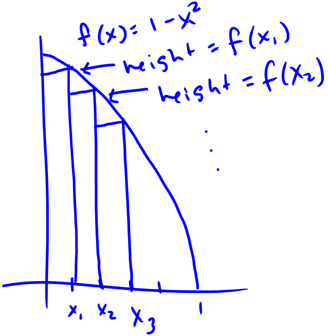


Let's take a partition w/ n subintervals.

$\Delta x = \frac{b-a}{n} = \frac{1}{n}$

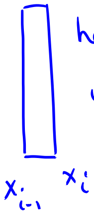
$x_0 = 0, \; x_1 = \frac{1}{n}, \; x_2 = \frac{2}{n}, \; x_3 = \frac{3}{n}, \ldots$

$x_i = \frac{i}{n}$



right end points

A representative rectangle



height $f(x_i) = 1 - x_i^2 = 1 - \left(\frac{i}{n}\right)^2$

width $\Delta x = \frac{1}{n}$

Area of one rectangle = height × width

$f(x_i)\,\Delta x = \left(1 - \frac{i^2}{n^2}\right)\cdot\frac{1}{n}$

Our Riemann sum is

$$f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(1 - \frac{i^2}{n^2}\right) \cdot \frac{1}{n}$$

We need to take the limit as $n \to \infty$.

Let's clean it up first:

$$\sum_{i=1}^{n} \left(1 - \frac{i^2}{n^2}\right) \cdot \frac{1}{n} = \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{i^2}{n^3}\right)$$

$$= \sum_{i=1}^{n} \frac{1}{n} - \sum_{i=1}^{n} \frac{i^2}{n^3}$$

split the difference

$$= \frac{1}{n} \sum_{i=1}^{n} 1 - \frac{1}{n^3} \sum_{i=1}^{n} i^2$$

factor out $\frac{1}{n}$ and $\frac{1}{n^3}$

$$= \frac{1}{n}(n) - \frac{1}{n^3}\left(\frac{2n^3+3n^2+n}{6}\right)$$

use the given formula and *

$$* \quad \sum_{i=1}^{n} 1 = \underbrace{1+1+1+\ldots+1}_{n \text{ of these}} = n$$

Our approximation is

$$A \approx 1 - \frac{2n^3 + 3n^2 + n}{6n^3}$$

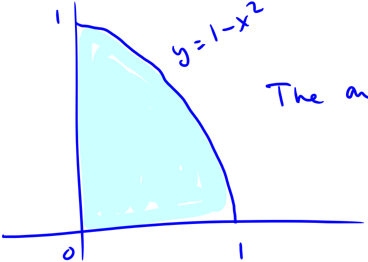
The true area is obtained by taking $n \to \infty$.

$$A = \lim_{n \to \infty} \left(1 - \frac{2n^3 + 3n^2 + n}{6n^3}\right)$$

$$= \lim_{n \to \infty} \left(1 - \frac{2n^3 + 3n^2 + n}{6n^3} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}}\right)$$

$$= \lim_{n\to\infty} \left(1 - \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{6} \right)$$

$$= 1 - \frac{2+0+0}{6} = 1 - \frac{2}{6} = 1 - \frac{1}{3} = \frac{2}{3}$$



The area shaded is $2/3$ square units.