Nov. 9 Math 1190 sec. 52 Fall 2016

Section 5.1: Area (under the graph of a nonnegative function)

We will investigate the area enclosed by the graph of a function f. We'll make the following assumptions (for now):

- f is continuous on the interval $[a, b]$, and
- f is nonnegative, i.e $f(x) \geq 0$, on $[a, b]$.

Our Goal: Find the area of such a region.

We'll start by approximating the region with a bunch of rectangles, then move to the exact value.

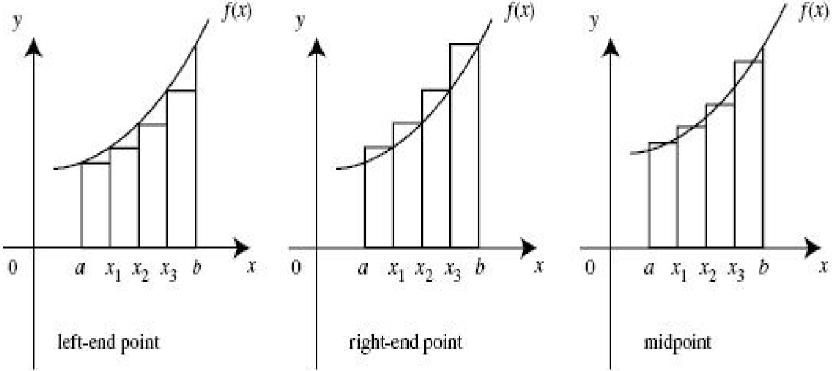


Figure: Some choices as to how to define the heights.

Approximating Area Using Rectangles

We can experiment with

- Which points to use for the heights (left, right, middle, other....)

- How many rectangles we use

to try to get a good approximation.

Definition: We will define the true area to be value we obtain taking the limit as the number of rectangles goes to $+\infty$.

Some terminology

- A **Partition** P of an interval $[a, b]$ is a collection of points $\{x_0, x_1, ..., x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

- A **Subinterval** is one of the intervals $x_{i-1} \leq x \leq x_i$ determined by a partition.
- The width of a subinterval is denoted $\Delta x_i = x_i - x_{i-1}$. If they are all the same size (equal spacing), then

$$\Delta x = \frac{b-a}{n}, \quad \text{and this is called the } \textbf{norm} \text{ of the partition.}$$

- A set of **sample points** is a set $\{c_1, c_2, \ldots, c_n\}$ such that $x_{i-1} \leq c_i \leq x_i$.

Taking the number of rectangles to ∞ is the same as taking the width $\Delta x \to 0$.

Example:

Write an equally spaced partition of the interval $[0, 2]$ with the specified number of subintervals, and determine the norm Δx.

(a) For $n = 4$

We found that $\Delta x = (2 - 0)/4 = 1/2$. So our partition points were

$$x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 1, \quad x_3 = \frac{3}{2}, \quad x_4 = 2$$

Note that we can write the partition points using a formula

$$x_i = 0 + i\Delta x = \frac{i}{2}.$$

(c) Find an equally spaced partition of $[0, 2]$ having N subintervals. What is the norm Δx?

$b = 2, \quad a = 0, \quad n = N$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{N} = \frac{2}{N}$$

$x_0 = 0$

$x_N = 2$

$x_1 = \frac{2}{N}$

$x_2 = 2 \cdot \frac{2}{N}$

$x_3 = 3 \cdot \frac{2}{N}$

$\vdots$

$$x_i = 0 + i\,\Delta x = i\left(\frac{2}{N}\right) = \frac{2i}{N}$$

$$\left\{ x_i \;\middle|\; x_i = i\left(\frac{2}{N}\right),\ i = 0, \ldots, N \right\}$$

Note $\quad x_N = \frac{2N}{N} = 2$ as required.

Approximating area with a Partition and sample points

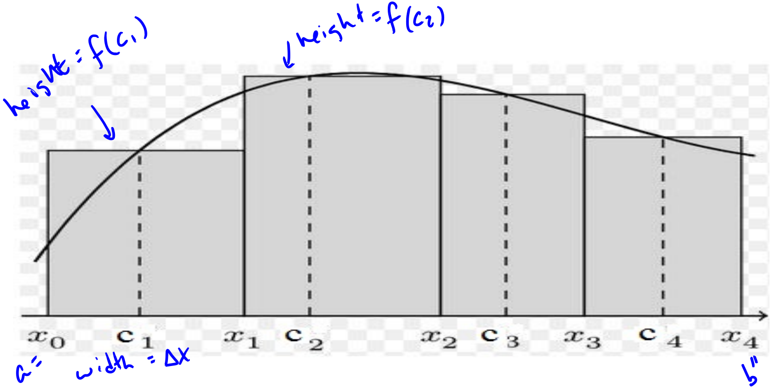


Figure: Area $\approx f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + f(c_4)\Delta x$. This can be written as

$$\sum_{i=1}^{n} f(c_i)\Delta x.$$

In general, an equally spaced partition of $[a, b]$ with n subintervals means

- $\Delta x = \frac{b-a}{n}$
- $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, i.e. $x_i = a + i\Delta x$
- Taking heights to be

$$\text{left ends} \quad c_i = x_{i-1} \quad \text{area} \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$\text{right ends} \quad c_i = x_i \quad \text{area} \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

- The true area exists (for f continuous) and is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

Lower and Upper Sums

The standard way to set up these sums is to take c_i such that

$$f(c_i) \quad \text{is the abs. minimum value of } f \text{ on } [x_{i-1}, x_i]$$

Then set A_L

$$A_L = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

This is called a **Lower Riemann sum**.

Lower and Upper Sums

Then, we take C_i such that

$$f(C_i) \quad \text{is the abs. maximum value of } f \text{ on } [x_{i-1}, x_i]$$

Then set A_U

$$A_U = \lim_{n\to\infty} \sum_{i=1}^{n} f(C_i)\Delta x.$$

This is called a **Upper Riemann sum**.

Lower and Upper Sums

If f is continuous on $[a, b]$, then it will necessarily be that

$$A_L = A_U.$$

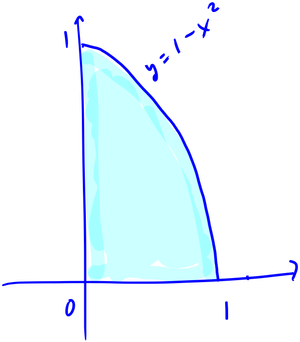
This value is the true area.

In practice, these are tough to compute unless f is only increasing or only decreasing. So instead, we tend to use left and right sums.

Example: Find the area under the curve $y = 1 - x^2$, $0 \leq x \leq 1$.

Use right end points $c_i = x_i$ and assume the following identity

$$\sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6} \quad \text{(sum of first } n \text{ squares)}$$



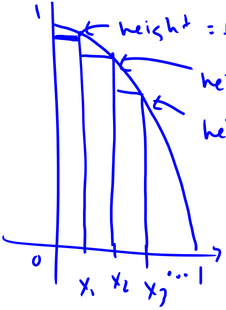
Let's introduce a partition of n subintervals.

norm: $\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

$x_0 = 0$, $x_1 = \frac{1}{n}$, $x_2 = \frac{2}{n}$, $x_3 = \frac{3}{n}$, ...

$x_i = \frac{i}{n}$

right end points

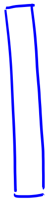


height $= f(x_1)$

height $= f(x_2)$

height $= f(x_3)$

representative rectangle



height $= f(x_i) = 1 - \left(\frac{i}{n}\right)^2$

width $= \Delta x = \frac{1}{n}$

area of one rectangle =

height $\times$ width $= f(x_i)\Delta x = \left(1 - \frac{i^2}{n^2}\right)\cdot\frac{1}{n}$

Adding n-rectangles

$$A \approx f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$$

$$= \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \left(1 - \frac{i^2}{n^2}\right) \cdot \frac{1}{n}$$

We want to take the limit $n \to \infty$.
Let's simplify the formula first.

$$A \approx \sum_{i=1}^{n} \left(1 - \frac{i^2}{n^2}\right) \cdot \frac{1}{n} = \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{i^2}{n^3}\right)$$

$$= \sum_{i=1}^{n} \frac{1}{n} - \sum_{i=1}^{n} \frac{i^2}{n^3}$$

$$= \frac{1}{n}\sum_{i=1}^{n} 1 - \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{1}{n}(n) - \frac{1}{n^3}\left(\frac{2n^3 + 3n^2 + n}{6}\right)$$

$$= 1 - \frac{2n^3 + 3n^2 + n}{6n^3}$$

$$* \quad \sum_{i=1}^{n} 1 = \underbrace{1 + 1 + 1 + \ldots + 1}_{n \text{ times}} = n$$

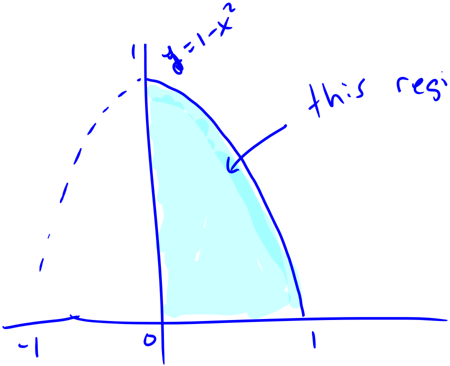
The true area is the value in the limit $n \to \infty$

$$A = \lim_{n\to\infty} \left(1 - \frac{2n^3 + 3n^2 + n}{6n^3}\right)$$

$$= \lim_{n\to\infty} \left(1 - \frac{2n^3 + 3n^2 + n}{6n^3} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}}\right)$$

$$= \lim_{n\to\infty} \left(1 - \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{6}\right)$$

$$= 1 - \frac{2+0+0}{6} = 1 - \frac{2}{6} = 1 - \frac{1}{3} = \frac{2}{3}$$



this region has area $\frac{2}{3}$ square units.

Recovering Distance from Velocity

The speedometer readings for a motorcycle are recorded at 12 second intervals. Use the information in the table to estimate the total distance traveled. Get estimates using
(a) left end points (beginning time of intervals), and
(b) right end points (ending time for each interval).

t in sec	0	12	24	36	48	60
v in ft/sec	20	28	25	22	24	27

distance = rate times time is like

area = height times width

t in sec	0	12	24	36	48	60
v in ft/sec	20	28	25	22	24	27

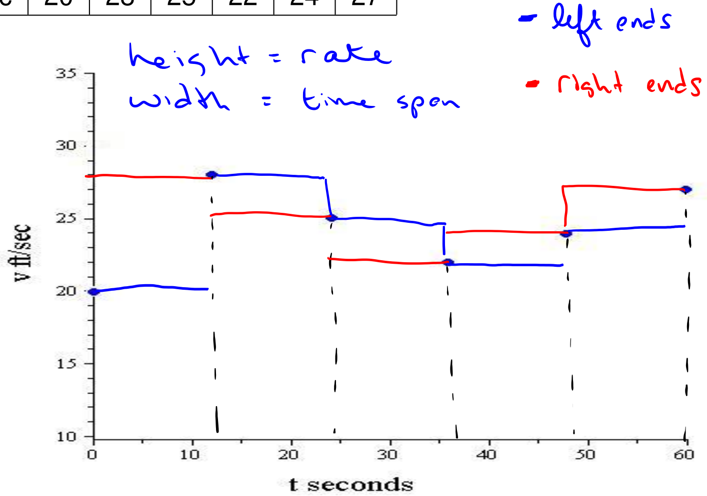


Figure: Graphical representation of motorcycle's velocity.

t in sec	0	12	24	36	48	60
v in ft/sec	20	28	25	22	24	27

Left end approximation call this D_L

$$D_L = 20\,\frac{\text{ft}}{\text{sec}}\cdot 12\,\text{sec} + 28\,\frac{\text{ft}}{\text{sec}}\cdot 12\,\text{sec} + 25\,\frac{\text{ft}}{\text{sec}}\cdot 12\,\text{sec}$$

$$+ 22\,\frac{\text{ft}}{\text{sec}}\cdot 12\,\text{sec} + 24\,\frac{\text{ft}}{\text{sec}}\cdot 12\,\text{sec}.$$