

### Section 7.3: Translation Theorems

**Theorem (translation in  $s$ )** Suppose  $\mathcal{L}\{f(t)\} = F(s)$ . Then for any real number  $a$

$$\mathcal{L}\{e^{at}f(t)\} = F(s - a).$$

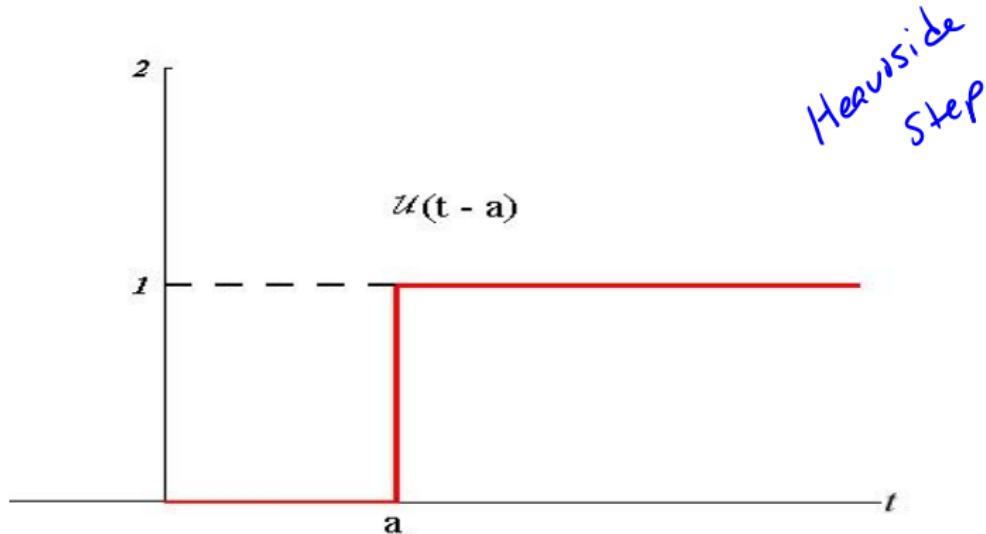
Consequently  $\mathcal{L}^{-1}\{F(s - a)\} = e^{at}f(t)$  if  $\mathcal{L}^{-1}\{F(s)\} = f(t)$ .

Next, we will consider a horizontal shift in  $t$ . Since we interested in functions defined on  $0 \leq t < \infty$ , we need a preliminary concept.

## The Unit Step Function

Let  $a \geq 0$ . The unit step function  $\mathcal{U}(t - a)$  is defined by

$$\mathcal{U}(t - a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$



## Example

Plot the graph of  $f(t) = \mathcal{U}(t - 1) - \mathcal{U}(t - 3)$ .

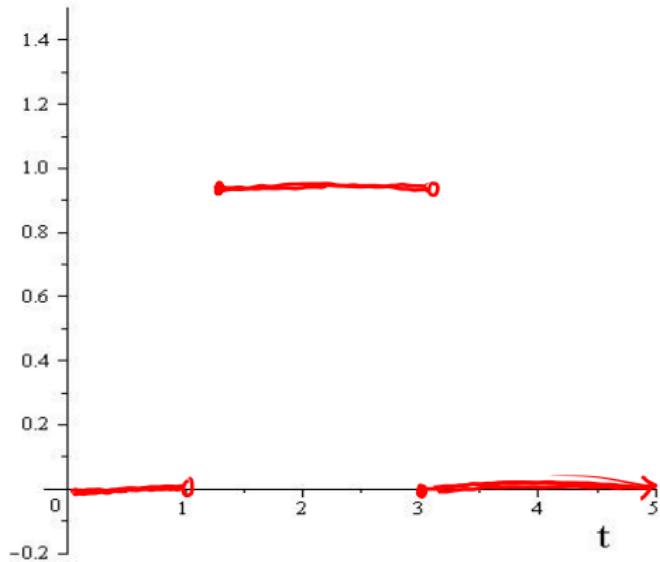
$$u(t-1) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & t \geq 1 \end{cases} \quad u(t-3) = \begin{cases} 0, & 0 \leq t < 3 \\ 1, & t \geq 3 \end{cases}$$

For  $0 \leq t < 1$   $f(t) = 0 - 0 = 0$

For  $1 \leq t < 3$   $f(t) = 1 - 0 = 1$

For  $t \geq 3$   $f(t) = 1 - 1 = 0$

Plot of  $f(t) = u(t-1) - u(t-3)$



## Piecewise Defined Functions

Verify that

$$f(t) = \begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases} = g(t) - g(t-a)U(t-a) + h(t)U(t-a)$$

For  $0 \leq t < a$ ,  $U(t-a) = 0$

$$f(t) = g(t) - g(t) \cdot 0 + h(t) \cdot 0 = g(t) \text{ as required}$$

For  $t \geq a$ ,  $U(t-a) = 1$

$$f(t) = g(t) - g(t) \cdot 1 + h(t) \cdot 1 = h(t) \text{ as required.}$$

## Piecewise Defined Functions

Express  $f$  on one line in terms of unit step functions.

$$f(t) = \begin{cases} 3t, & 0 \leq t < 1 \\ t^2, & 1 \leq t < 5 \\ e^t, & t \geq 5 \end{cases}$$

$$f(t) = 3t - 3tU(t-1) + t^2U(t-1) - t^2U(t-5) + e^tU(t-5)$$

turn off      turn on      turn off      turn on

We can double check for accuracy:

$$\text{For } 0 \leq t < 1 \quad U(t-1) = 0 \quad \text{and} \quad U(t-5) = 0$$

$$\text{then } f(t) = 3t - 3t \cdot 0 + t^2 \cdot 0 - t^2 \cdot 0 + e^t \cdot 0 = 3t$$

$$\text{For } 1 \leq t < 5, \quad u(t-1) = 1, \quad u(t-5) = 0$$

$$\text{then } f(t) = 3t - 3t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 0 + e^t \cdot 0 = t^2$$

$$\text{For } t \geq 5, \quad u(t-1) = 1 \quad u(t-5) = 1$$

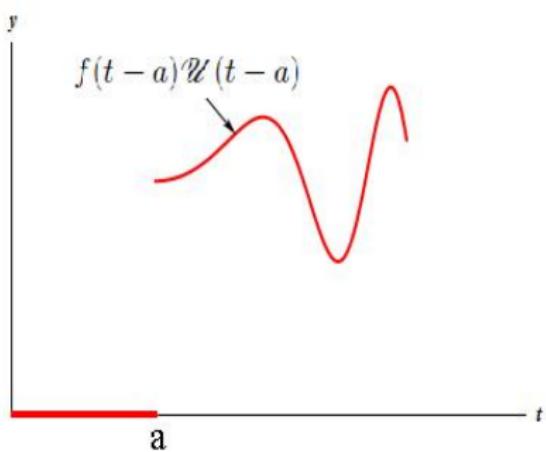
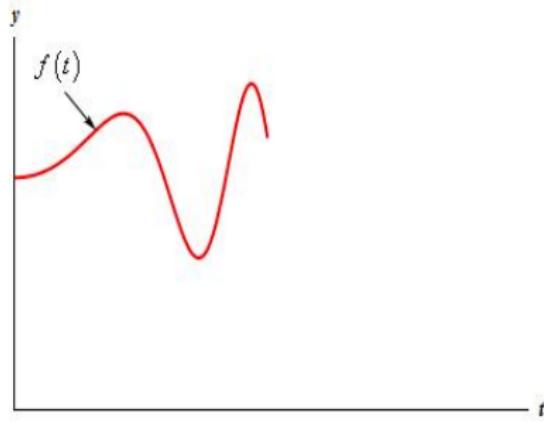
$$\text{then } f(t) = 3t - 3t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 1 + e^t \cdot 1 = e^t$$

as required

## Translation in $t$

Given a function  $f(t)$  for  $t \geq 0$ , and a number  $a > 0$

$$f(t - a)\mathcal{U}(t - a) = \begin{cases} 0, & 0 \leq t < a \\ f(t - a), & t \geq a \end{cases} .$$



## Theorem (translation in $t$ )

If  $F(s) = \mathcal{L}\{f(t)\}$  and  $a > 0$ , then

$$\mathcal{L}\{f(t - a)\mathcal{U}(t - a)\} = e^{-as}F(s).$$

In particular,

$$\mathcal{L}\{\mathcal{U}(t - a)\} = \frac{e^{-as}}{s}.$$

As another example,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \implies \mathcal{L}\{(t - a)^n\mathcal{U}(t - a)\} = \frac{n!e^{-as}}{s^{n+1}}.$$

## Example

Find the Laplace transform  $\mathcal{L}\{f(t)\}$  where

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ t, & t \geq 1 \end{cases} = 1 - 1\mathcal{U}(t-1) + t\mathcal{U}(t-1)$$

$$= 1 + (t-1)\mathcal{U}(t-1)$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{1 + (t-1)\mathcal{U}(t-1)\} \\ &= \mathcal{L}\{1\} + \mathcal{L}\{(t-1)\mathcal{U}(t-1)\}\end{aligned}$$

$$= \frac{1}{s} + \frac{1}{s^2} e^{-s}$$

## A Couple of Useful Results

Another formulation of this translation theorem is

$$(1) \quad \mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}.$$

Note  $g(t) = g(t - a + a)$

The inverse form of this translation theorem is

$$(2) \quad \mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a).$$

## Evaluate the Laplace or Inverse Laplace Transform

$$\begin{aligned}(a) \quad \mathcal{L}\{2t\mathcal{U}(t-3)\} &= \mathcal{L}\{2(t+3)\} e^{-3s} \\&= e^{-3s} \mathcal{L}\{2t+6\} \\&= e^{-3s} \left( 2\mathcal{L}\{t\} + 6\mathcal{L}\{1\} \right) \\&= e^{-3s} \left( \frac{2}{s^2} + \frac{6}{s} \right) = \frac{2e^{-3s}}{s^2} + \frac{6e^{-3s}}{s}\end{aligned}$$

$$* \quad 2t = 2(t-3+3) = 2(t-3)+6$$

$$\begin{aligned}
 (b) \quad \mathcal{L}\{\cos t \mathcal{U}\left(t - \frac{\pi}{2}\right)\} &= e^{-\frac{\pi}{2}s} \mathcal{L}\left\{\cos\left(t + \frac{\pi}{2}\right)\right\} \\
 &= e^{-\frac{\pi}{2}s} \mathcal{L}\{-\sin t\} \\
 &= -e^{-\frac{\pi}{2}s} \mathcal{L}\{\sin t\} = \frac{-e^{-\frac{\pi}{2}s}}{s^2 + 1}
 \end{aligned}$$

$$* \cos\left(t + \frac{\pi}{2}\right) = \cos t \cos \frac{\pi}{2} - \sin t \sin \frac{\pi}{2} = -\sin t$$

$$(c) \quad \mathcal{L}^{-1} \left\{ \frac{e^{-4s}}{s^3} \right\}$$

we need  $\mathcal{L}^{-1} \left\{ \frac{1}{s^3} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{2!} \frac{s^2}{s^3} \right\}$

$$= \frac{1}{2!} \mathcal{L}^{-1} \left\{ \frac{s^2}{s^3} \right\}$$

$$= \frac{1}{2!} (t-4)^2 u(t-4)$$

$$= \frac{1}{2!} t^2$$

$$(d) \quad \mathcal{L}^{-1} \left\{ \frac{e^{-s}}{s(s+1)} \right\}$$

we need  $\mathcal{L}^{-1} \left\{ \frac{1}{s(s+1)} \right\}$

Decomp

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$1 = A(s+1) + Bs$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{1}{s+1} \right\}$$

$$\text{set } s=0 \quad 1=A$$

$$s=-1 \quad 1=-B$$

$$= 1 - e^{-t}$$

$$\mathcal{L}^{-1} \left\{ \frac{e^{-s}}{s(s+1)} \right\} = 1 u(t-1) - e^{-(t-1)} u(t-1)$$