November 9 Math 2306 sec. 53 Fall 2018

Section 16: Laplace Transforms of Derivatives and IVPs

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\left\{y(t)\right\} = Y(s),$$

then

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0),$$

$$\vdots$$

$$\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).$$

Solving Constant Coefficient IVPs

Consider the constant coefficient ODE with initial conditions given at $t = 0$.

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

The Laplace transform

$$Y(s) = \mathscr{L}\left\{y(t)\right\} = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where $G(s) = \mathscr{L}\left\{g(t)\right\}$ and P is the characteristic polynomial for the ODE.

The solution to the IVP satisfies $y(t) = \mathscr{L}^{-1}\left\{Y(s)\right\}$.

Zero Input/Zero State Responses

In the solution to the IVP

$$y(t) = \mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\} + \mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$$

$\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}$ is called the **zero input response,**

and

$\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$ is called the **zero state response.**

Solve the IVP using the Laplace Transform

(a) $\dfrac{dy}{dt} + 3y = 2t \quad y(0) = 2$

Take transform of ODE

Let $Y(s) = \mathcal{L}\{y(t)\}$

$\mathcal{L}\{y' + 3y\} = \mathcal{L}\{2t\}$

$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 2\mathcal{L}\{t\}$

$sY(s) - y(0) + 3Y(s) = \dfrac{2}{s^2}$

Solve for $Y(s)$

$(s+3)Y(s) - 2 = \dfrac{2}{s^2}$

$$(s+3)Y(s) = \frac{2}{s^2} + 2$$

$$Y(s) = \frac{2}{s^2(s+3)} + \frac{2}{s+3}$$

We'll do a partial fraction decomp on $\frac{2}{s^2(s+3)}$

$$\frac{2}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

Multiply by $s^2(s+3)$

$$2 = As(s+3) + B(s+3) + Cs^2$$

$$= A(s^2+3s) + B(s+3) + Cs^2$$

$$0s^2 + 0s + 2 = (A+C)s^2 + (3A+B)s + 3B$$

$A + C = 0 \qquad \Rightarrow \quad C = -A$

$3A + B = 0 \quad \Rightarrow \quad A = \frac{-1}{3} B$

$3B = 2$

$B = \frac{2}{3}, \quad A = \frac{-2}{9}, \quad C = \frac{2}{9}$

$$Y(s) = \frac{\frac{-2}{9}}{s} + \frac{\frac{2}{3}}{s^2} + \frac{\frac{2}{9}}{s+3} + \frac{2}{s+3}$$

$$Y(s) = \frac{\frac{-2}{9}}{s} + \frac{\frac{2}{3}}{s^2} + \frac{20/9}{s+3}$$

The solution to the IVP

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{-2/9}{s} + \frac{2/3}{s^2} + \frac{20/9}{s+3} \right\}$$

$$= \frac{-2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{20}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$\boxed{y(t) = \frac{-2}{9} + \frac{2}{3}t + \frac{20}{9}e^{-3t}}$$

Let's verify y solves

$$y' + 3y = 2t \qquad y(0) = 2$$

$$y(0) = \frac{-2}{9} + \frac{2}{3}\cdot 0 + \frac{20}{9}\cdot e^{0} = \frac{-2}{9} + \frac{20}{9} = \frac{18}{9} = 2$$

$$y' = \frac{2}{3} + \frac{20}{9}(-3)e^{-3t} = \frac{2}{3} - \frac{20}{3}e^{-3t}$$

So

$$y' + 3y = \frac{2}{3} - \frac{20}{3}e^{-3t} + 3\left(\frac{-2}{9} + \frac{2}{3}t + \frac{20}{9}e^{-3t}\right)$$

$$= \frac{2}{3} - \frac{20}{3}e^{-3t} - \frac{2}{3} + 2t + \frac{20}{3}e^{-3t}$$

$$= 2t \quad \text{as required.}$$

Solve the IVP using the Laplace Transform

$$y'' + 4y' + 4y = te^{-2t} \quad y(0) = 1, y'(0) = 0$$

Let $Y(s) = \mathcal{L}\{y(t)\}$

$\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{te^{-2t}\}$

$\mathcal{L}\{f(t)e^{at}\} = F(s-a)$

$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \dfrac{1}{(s+2)^2}$

$\mathcal{L}\{t\} = \dfrac{1}{s^2}$

and $a = -2$

$s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 4Y(s) = \dfrac{1}{(s+2)^2}$

$y(0) \downarrow 1 \quad y'(0) \downarrow 0 \quad y(0) \downarrow 1$

$$\underbrace{(s^2+4s+4)}Y(s) - s - 4 = \frac{1}{(s+2)^2}$$

Characteristic polynomial for

$$y'' + 4y' + 4y = 0$$

We'll finish next time.