

### Section 7.3: Translation Theorems

**Theorem (translation in  $s$ )** Suppose  $\mathcal{L}\{f(t)\} = F(s)$ . Then for any real number  $a$

$$\mathcal{L}\{e^{at}f(t)\} = F(s - a).$$

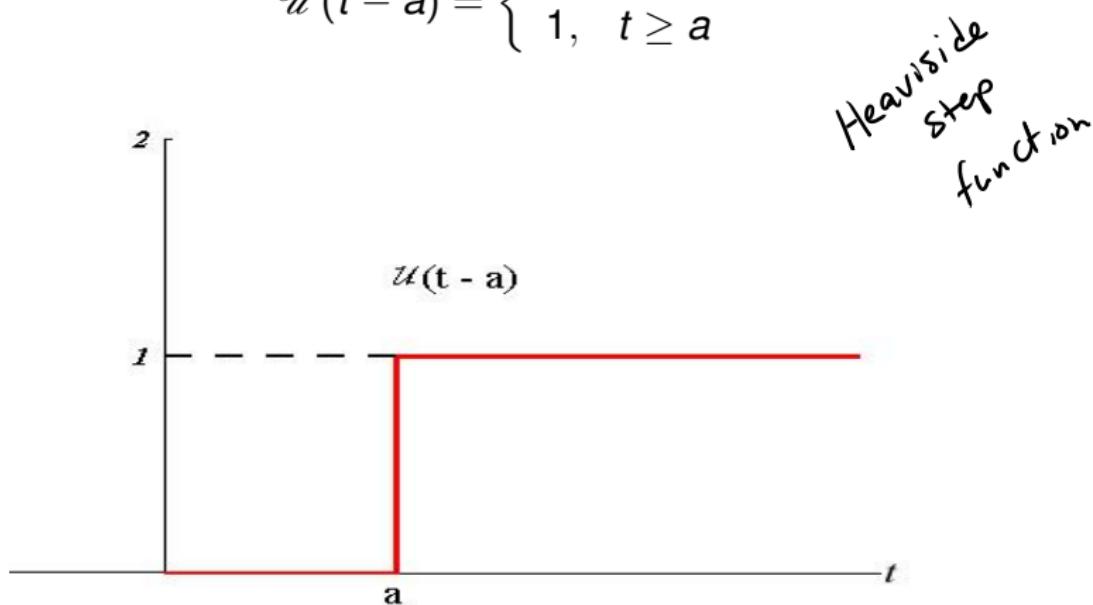
Consequently  $\mathcal{L}^{-1}\{F(s - a)\} = e^{at}f(t)$  if  $\mathcal{L}^{-1}\{F(s)\} = f(t)$ .

Next, we will consider a horizontal shift in  $t$ . Since we interested in functions defined on  $0 \leq t < \infty$ , we need a preliminary concept.

# The Unit Step Function

Let  $a \geq 0$ . The unit step function  $\mathcal{U}(t - a)$  is defined by

$$\mathcal{U}(t - a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$



## Example

Plot the graph of  $f(t) = \mathcal{U}(t - 1) - \mathcal{U}(t - 3)$ .

$$u(t-1) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & t \geq 1 \end{cases} \quad u(t-3) = \begin{cases} 0, & 0 \leq t < 3 \\ 1, & t \geq 3 \end{cases}$$

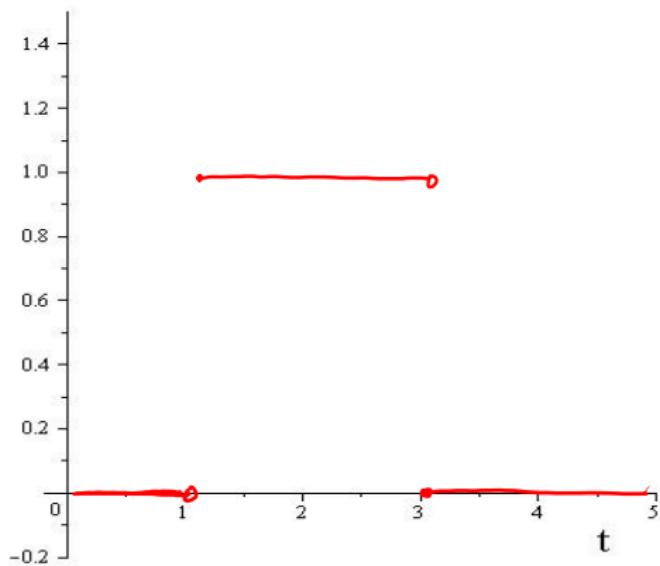
so for  $0 \leq t < 1$ , For  $1 \leq t < 3$

$$f(t) = 0 - 0 = 0 \quad f(t) = 1 - 0 = 1$$

and for

$$t \geq 3 \quad f(t) = 1 - 1 = 0$$

$$\text{Plot } f(t) = u(t-1) - u(t-3)$$



## Piecewise Defined Functions

Verify that

$$f(t) = \begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases} = g(t) - g(t-a)U(t-a) + h(t)U(t-a)$$

If  $0 \leq t < a$ , then  $U(t-a) = 0$

$$\text{so } f(t) = g(t) - g(t) \cdot 0 + h(t) \cdot 0 = g(t)$$

If  $t > a$ , then  $U(t-a) = 1$  so that

$$f(t) = g(t) - g(t) \cdot 1 + h(t) \cdot 1 = h(t)$$

For all  $t$ , we get the correct  $f(t)$ .

## Piecewise Defined Functions

Express  $f$  on one line in terms of unit step functions.

$$f(t) = \begin{cases} 3t, & 0 \leq t < 1 \\ t^2, & 1 \leq t < 5 \\ e^t, & t \geq 5 \end{cases}$$

$$= 3t - 3tU(t-1) + t^2U(t-1) - t^2U(t-5) + e^tU(t-5)$$

We can verify that this is correct.

Note: If  $0 \leq t < 1$ ,  $U(t-1) = 0$  and  $U(t-5) = 0$

$$f(t) = 3t - 3t \cdot 0 + t^2 \cdot 0 - t^2 \cdot 0 + e^t \cdot 0 = 3t$$

as required

If  $1 \leq t < 5$ ,  $u(t-1) = 1$  and  $u(t-5) = 0$

$$f(t) = 3t - 3t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 0 + e^t \cdot 0 = t^2$$

as required

And if  $t \geq 5$ ,  $u(t-1) = 1$  and  $u(t-5) = 1$

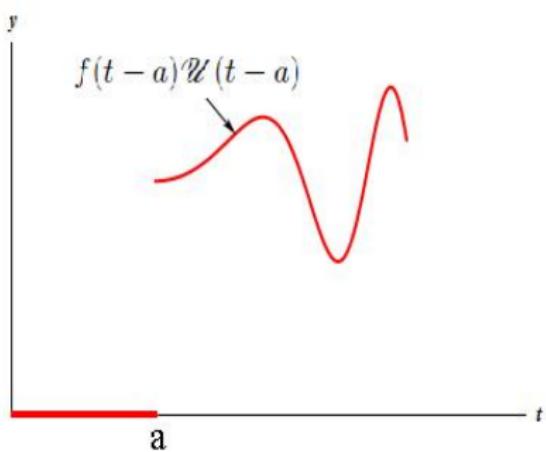
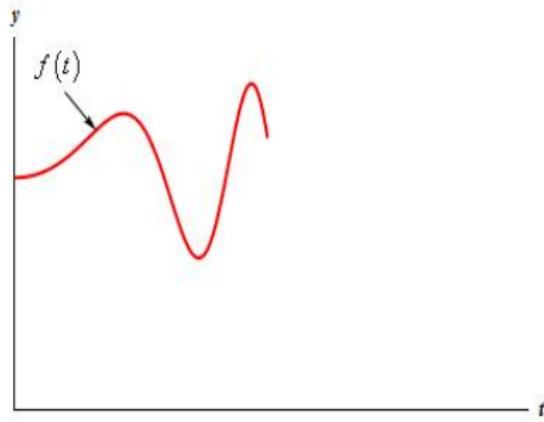
$$f(t) = 3t - 3t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 1 + e^t \cdot 1 = e^t$$

again as it should be.

## Translation in $t$

Given a function  $f(t)$  for  $t \geq 0$ , and a number  $a > 0$

$$f(t - a)\mathcal{U}(t - a) = \begin{cases} 0, & 0 \leq t < a \\ f(t - a), & t \geq a \end{cases} .$$



## Theorem (translation in $t$ )

If  $F(s) = \mathcal{L}\{f(t)\}$  and  $a > 0$ , then

$$\mathcal{L}\{f(t - a)\mathcal{U}(t - a)\} = e^{-as}F(s).$$

In particular,

$$\mathcal{L}\{\mathcal{U}(t - a)\} = \frac{e^{-as}}{s}.$$

As another example,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \implies \mathcal{L}\{(t - a)^n\mathcal{U}(t - a)\} = \frac{n!e^{-as}}{s^{n+1}}.$$

## Example

Find the Laplace transform  $\mathcal{L}\{f(t)\}$  where

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ t, & t \geq 1 \end{cases} = 1 - 1\mathcal{U}(t-1) + t\mathcal{U}(t-1)$$

$$= 1 + (t-1)\mathcal{U}(t-1)$$

note  
 $\mathcal{L}\{t\} = \frac{1}{s^2}$

$$\begin{aligned} \text{so } \mathcal{L}\{f(t)\} &= \mathcal{L}\{1 + (t-1)\mathcal{U}(t-1)\} \\ &= \mathcal{L}\{1\} + \mathcal{L}\{(t-1)\mathcal{U}(t-1)\} \\ &= \frac{1}{s} + \frac{e^{-s}}{s^2} \end{aligned}$$

# A Couple of Useful Results

Another formulation of this translation theorem is

$$(1) \quad \mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}.$$

Note that

$$g(t) = g(t-a+a)$$

The inverse form of this translation theorem is

$$(2) \quad \mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a).$$