November 9 Math 2306 sec. 56 Fall 2017

Section 17: Fourier Series: Trigonometric Series

Some Preliminary Concepts

Suppose two functions f and g are integrable on the interval $[a, b]$. We define the **inner product** of f and g on $[a, b]$ as

$$< f, g >= \int_a^b f(x)g(x)\,dx.$$

We say that f and g are **orthogonal** on $[a, b]$ if

$$< f, g >= 0.$$

The product depends on the interval, so the orthogonality of two functions depends on the interval.

Properties of an Inner Product

Let f, g, and h be integrable functions on the appropriate interval and let c be any real number. The following hold

(i) $<f, g>=<g, f>$

(ii) $<f, g+h>=<f, g>+<f, h>$

(iii) $<cf, g>=c<f, g>$

(iv) $<f, f>\geq 0$ and $<f, f>=0$ if and only if $f=0$

Orthogonal Set

A set of functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be **orthogonal** on an interval $[a, b]$ if

$$< \phi_m, \phi_n >= \int_a^b \phi_m(x)\phi_n(x)\,dx = 0 \quad \text{whenever} \quad m \neq n.$$

Note that any function $\phi(x)$ that is not identically zero will satisfy

$$< \phi, \phi >= \int_a^b \phi^2(x)\,dx > 0.$$

Hence we define the **square norm** of ϕ (on $[a, b]$) to be

$$\|\phi\| = \sqrt{\int_a^b \phi^2(x)\,dx}.$$

An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\} \quad \text{on} \quad [-\pi, \pi].$$

$[a, b]$

Evaluate $\langle \cos(nx), 1 \rangle$ and $\langle \sin(mx), 1 \rangle$ for $n, m \geq 1$.

$$\langle \cos(nx), 1 \rangle = \int_{-\pi}^{\pi} \cos(nx) \cdot 1 \, dx$$

$$= \int_{-\pi}^{\pi} \cos(nx)\, dx = \frac{1}{n} \sin(nx) \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{n} \sin(n\pi) - \frac{1}{n} \sin(-n\pi) = 0 - 0 = 0$$

$$\langle \sin(mx), 1\rangle = \int_{-\pi}^{\pi} \sin(mx)\cdot 1 \, dx$$

$$= \int_{-\pi}^{\pi} \sin(mx)\, dx$$

$$= \frac{-1}{m}\cos(mx)\Big|_{-\pi}^{\pi}$$

$$= \frac{-1}{m}\cos(m\pi) - \frac{-1}{m}\cos(-m\pi) \qquad \cos(-\theta) = \cos\theta$$

$$= \frac{-1}{m}\cos(m\pi) + \frac{1}{m}\cos(m\pi)$$

$$= 0$$

An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\} \quad \text{on} \quad [-\pi, \pi].$$

It can easily be verified that

$$\int_{-\pi}^{\pi} \cos nx \, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin mx \, dx = 0 \quad \text{for all} \quad n, m \geq 1,$$

$$\int_{-\pi}^{\pi} \cos nx \, \sin mx \, dx = 0 \quad \text{for all} \quad m, n \geq 1, \quad \text{and}$$

$$\int_{-\pi}^{\pi} \cos nx \, \cos mx \, dx = \int_{-\pi}^{\pi} \sin nx \, \sin mx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & n = m \end{cases},$$

An Orthogonal Set of Functions on $[-\pi, \pi]$

These integral values indicated that the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\}$$

is an orthogonal set on the interval $[-\pi, \pi]$.

An Orthogonal Set of Functions on $[-p, p]$

This set can be generalized by using a simple change of variables $t = \frac{\pi x}{p}$ to obtain the orthogonal set on $[-p, p]$

$$\left\{1, \cos\frac{n\pi x}{p}, \sin\frac{m\pi x}{p} \,|\, n, m \in \mathbb{N}\right\}$$

There are many interesting and useful orthogonal sets of functions (on appropriate intervals). What follows is readily extended to other such (infinite) sets.

$\in$ means "is an element of"

$\mathbb{N}$ - set of natural numbers 1, 2, 3, 4, ...

Fourier Series

Suppose $f(x)$ is defined for $-\pi < x < \pi$. We would like to know how to write f as a series **in terms of sines and cosines.**

Task: Find coefficients (numbers) a_0, $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that[1]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right).$$

[1] We'll write $\frac{a_0}{2}$ as opposed to a_0 purely for convenience.

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,.$$

The question of convergence naturally arises when we wish to work with infinite series. To highlight convergence considerations, some authors prefer not to use the equal sign when expressing a Fourier series and instead write

$$f(x) \sim \frac{a_0}{2} + \cdots$$

Herein, we'll use the equal sign with the understanding that equality may not hold at each point.

Convergence will be address later.

Finding an Example Coefficient

For a known function f defined on $(-\pi, \pi)$, assume there is such a series[2]. Let's find the coefficient b_4.

$$f(x)\sin(4x) = \frac{a_0}{2}\sin(4x) + \sum_{n=1}^{\infty}\left(a_n \cos nx\sin(4x) + b_n \sin nx\sin(4x)\right).$$

Integrate both sides from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx =$$

$$\int_{-\pi}^{\pi}\left(\frac{a_0}{2}\sin(4x) + \sum_{n=1}^{\infty} a_n\cos(nx)\sin(4x) + b_n\sin(nx)\sin(4x)\right)dx$$

[2]We will also assume that the order of integrating and summing can be interchanged.

$$= \frac{a_0}{2}\int_{-\pi}^{\pi} \sin(4x)\,dx + \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos(nx)\sin(4x)\,dx + b_n\int_{-\pi}^{\pi}\sin(nx)\sin(4x)\,dx\right)$$

$\langle \sin(4x), 1\rangle = 0$

$\langle \cos(nx), \sin(4x)\rangle = 0$

$$\langle \sin(nx), \sin(4x)\rangle = \begin{cases} 0, & n\neq 4 \\ \pi, & n=4 \end{cases}$$

All terms except the one w/ b_4 are zero.

$$\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx = b_4(\pi)$$

$$\Rightarrow \quad b_4 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(4x) \, dx$$

Finding Fourier Coefficients

Note that there was nothing special about seeking the 4^{th} sine coefficient b_4. We could have just as easily sought b_m for any positive integer m. We would simply start by introducing the factor $\sin(mx)$.

Moreover, using the same orthogonality property, we could pick on the a's by starting with the factor $\cos(mx)$—including the constant term since $\cos(0\cdot x)=1$. The only minor difference we want to be aware of is that

$$\int_{-\pi}^{\pi}\cos^2(mx)\,dx=\begin{cases}2\pi, & m=0\\ \pi, & m\geq 1\end{cases}$$

Careful consideration of this sheds light on why it is conventional to take the constant to be $\frac{a_0}{2}$ as opposed to just a_0.

The Fourier Series of $f(x)$ on $(-\pi, \pi)$

The **Fourier series** of the function f defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

Where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad \text{and} \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}$$

Some Useful Observations

Integer multiples of π:

- For every integer n, $\sin(n\pi) = 0$,
- and $\cos(n\pi) = 1$ if n is even and $\cos(n\pi) = -1$ if n is odd. Thus we write
$$\cos(n\pi) = (-1)^n.$$

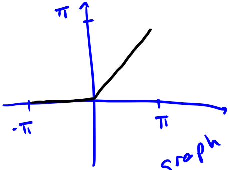
Symmetry:

- The sine function is odd, $\sin(-\theta) = -\sin(\theta)$,
- and the cosine function is even, $\cos(-\theta) = \cos(\theta)$.

Example

Find the Fourier series of the piecewise defined function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$$



graph of f

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \int_{-\pi}^{0} \cancelto{0}{0\,dx} + \frac{1}{\pi} \int_{0}^{\pi} x\,dx$$

$$= \frac{1}{\pi} \left. \frac{x^2}{2} \right|_0^{\pi} = \frac{1}{\pi} \left(\frac{\pi^2}{2} - \frac{0^2}{2} \right) = \frac{\pi^2}{2\pi} = \frac{\pi}{2}$$

$$\boxed{a_0 = \frac{\pi}{2}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \overbrace{0\cdot\cos(nx)\,dx}^{0} + \frac{1}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

Parts

$u = x$ $\quad du = dx$

$v = \frac{1}{n}\sin(nx)$ $\quad dv = \cos(nx)\,dx$

$$= \frac{1}{\pi}\left[\frac{x}{n}\sin(nx)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[\underbrace{\frac{\pi}{n}\sin(n\pi)}_{0} - \underbrace{\frac{0}{n}\sin(0)}_{0} + \frac{1}{n^2}\cos(nx)\Big|_0^{\pi}\right.$$

$$= \frac{1}{\pi}\left(\frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}\cos(0)\right)$$

$$\boxed{a_n = \frac{1}{\pi n^2}\left((-1)^n - 1\right)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \cancel{0\cdot\sin(nx)\,dx}^{\,0} + \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx$$

Parts

$u = x \qquad du = dx$

$v = \frac{-1}{n}\cos(nx) \qquad dv = \sin(nx)\,dx$

$$= \frac{1}{\pi}\left[\frac{-x}{n}\cos(nx)\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right.$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n}\cos(n\pi) - \frac{0}{n}\cos(0) + \frac{1}{n^2}\sin(nx)\Big|_0^{\pi}\right.$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n}(-1)^n + \frac{1}{n^2}\cancel{\sin(n\pi)}^{\,0} - \frac{1}{n^2}\cancel{\sin(0)}^{\,0}\right]$$

$$= \frac{1}{\pi}\left(\frac{-\pi}{n}\right)(-1)^n = \frac{-(-1)^n}{n} = \frac{(-1)^{n+1}}{n}$$

$$\boxed{b_n = \frac{(-1)^{n+1}}{n}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{1}{\pi n^2}\left((-1)^n - 1\right) \cos(nx) + \frac{(-1)^{n+1}}{n} \sin(nx) \right)$$