Suppose two functions $f$ and $g$ are integrable on the interval $[a, b]$. We define the \textbf{inner product} of $f$ and $g$ on $[a, b]$ as

$$\langle f, g \rangle = \int_{a}^{b} f(x)g(x) \, dx.$$ 

We say that $f$ and $g$ are \textbf{orthogonal} on $[a, b]$ if

$$\langle f, g \rangle = 0.$$

The product depends on the interval, so the orthogonality of two functions depends on the interval.
Properties of an Inner Product

Let $f$, $g$, and $h$ be integrable functions on the appropriate interval and let $c$ be any real number. The following hold

(i) $\langle f, g \rangle = \langle g, f \rangle$

(ii) $\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle$

(iii) $\langle cf, g \rangle = c \langle f, g \rangle$

(iv) $\langle f, f \rangle \geq 0$ and $\langle f, f \rangle = 0$ if and only if $f = 0$
Orthogonal Set

A set of functions \( \{ \phi_0(x), \phi_1(x), \phi_2(x), \ldots \} \) is said to be orthogonal on an interval \([a, b]\) if

\[
< \phi_m, \phi_n > = \int_a^b \phi_m(x)\phi_n(x) \, dx = 0 \quad \text{whenever} \quad m \neq n.
\]

Note that any function \( \phi(x) \) that is not identically zero will satisfy

\[
< \phi, \phi > = \int_a^b \phi^2(x) \, dx > 0.
\]

Hence we define the square norm of \( \phi \) (on \([a, b]\)) to be

\[
\| \phi \| = \sqrt{\int_a^b \phi^2(x) \, dx}.
\]
An Orthogonal Set of Functions

Consider the set of functions

\[ \{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\} \text{ on } \left[-\pi, \pi\right]. \]

Evaluate \( \langle \cos(nx), 1 \rangle \) and \( \langle \sin(mx), 1 \rangle \) for \( n, m \geq 1 \).

\[
\langle \cos(nx), 1 \rangle = \int_{-\pi}^{\pi} \cos(nx) \cdot 1 \, dx
\]

\[
= \int_{-\pi}^{\pi} \cos(nx) \, dx = \frac{1}{n} \sin(n \pi) \bigg|_{-\pi}^{\pi}
\]

\[
= \frac{1}{n} \sin(n \pi) - \frac{1}{n} \sin(-n \pi) = 0 - 0 = 0
\]
\[ \langle \sin(mx), 1 \rangle = \int_{-\pi}^{\pi} \sin(mx) \cdot 1 \, dx \]

\[ = \int_{-\pi}^{\pi} \sin(mx) \, dx \]

\[ = \frac{-1}{m} \cos(mx) \bigg|_{-\pi}^{\pi} \]

\[ = -\frac{1}{m} \cos(m\pi) - \frac{-1}{m} \cos(-m\pi) \]

\[ = -\frac{1}{m} \cos(m\pi) + \frac{1}{m} \cos(m\pi) \]

\[ = 0 \]
An Orthogonal Set of Functions

Consider the set of functions
\[
\{ 1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots \} \quad \text{on} \quad [-\pi, \pi].
\]

It can easily be verified that
\[
\int_{-\pi}^{\pi} \cos nx \, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin mx \, dx = 0 \quad \text{for all} \quad n, m \geq 1,
\]
\[
\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0 \quad \text{for all} \quad m, n \geq 1, \quad \text{and}
\]
\[
\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & n = m \end{cases}
\]
An Orthogonal Set of Functions on $[-\pi, \pi]$

These integral values indicated that the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\}$$

is an orthogonal set on the interval $[-\pi, \pi]$. 
An Orthogonal Set of Functions on $[-p, p]$

This set can be generalized by using a simple change of variables $t = \frac{\pi x}{p}$ to obtain the orthogonal set on $[-p, p]$

$$\left\{ 1, \cos \frac{n\pi x}{p}, \sin \frac{m\pi x}{p} \mid n, m \in \mathbb{N} \right\}$$

There are many interesting and useful orthogonal sets of functions (on appropriate intervals). What follows is readily extended to other such (infinite) sets.

$\in$ means "is an element of"

$\mathbb{N} -$ set of natural numbers $1, 2, 3, 4, \ldots$
Fourier Series

Suppose $f(x)$ is defined for $-\pi < x < \pi$. We would like to know how to write $f$ as a series in terms of sines and cosines.

**Task:** Find coefficients (numbers) $a_0$, $a_1$, $a_2$, ... and $b_1$, $b_2$, ... such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

\^We’ll write $\frac{a_0}{2}$ as opposed to $a_0$ purely for convenience.
Fourier Series

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) . \]

The question of convergence naturally arises when we wish to work with infinite series. To highlight convergence considerations, some authors prefer not to use the equal sign when expressing a Fourier series and instead write

\[ f(x) \sim \frac{a_0}{2} + \cdots \]

Herein, we’ll use the equal sign with the understanding that equality may not hold at each point.

Convergence will be address later.
Finding an Example Coefficient

For a known function $f$ defined on $(-\pi, \pi)$, assume there is such a series\(^2\). Let's find the coefficient $b_4$.

\[
f(x) \sin(yx) = \frac{a_0}{2} \sin(yx) + \sum_{n=1}^{\infty} \left( a_n \cos nx \sin(yx) + b_n \sin nx \sin(yx) \right).
\]

Integrate both sides from $-\pi$ to $\pi$.

\[
\int_{-\pi}^{\pi} f(x) \sin(yx) \, dx = 
\int_{-\pi}^{\pi} \left( \frac{a_0}{2} \sin(yx) + \sum_{n=1}^{\infty} a_n \cos(nx) \sin(yx) + b_n \sin(nx) \sin(yx) \right) \, dx
\]

\(^2\)We will also assume that the order of integrating and summing can be interchanged.
\[
\begin{align*}
\frac{a_0}{2} \int_{-\pi}^{\pi} \sin(4x) \, dx + \sum_{n=1}^{\infty} \left( a_n \int_{-\pi}^{\pi} \cos(nx) \sin(4x) \, dx + b_n \int_{-\pi}^{\pi} \sin(nx) \sin(4x) \, dx \right) \\
\langle \sin(4x), 1 \rangle = 0 \\
\langle \cos(nx), \sin(4x) \rangle = 0 \\
\langle \sin(nx), \sin(4x) \rangle = \begin{cases} 0, & n \neq 4 \\ \pi, & n = 4 \end{cases}
\end{align*}
\]

All terms except the one with \( b_4 \) are zero.

\[
\int_{-\pi}^{\pi} f(x) \sin(4x) \, dx = b_4 (\pi)
\]
\[ b_y = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(4x) \, dx \]
Finding Fourier Coefficients

Note that there was nothing special about seeking the $4^{th}$ sine coefficient $b_4$. We could have just as easily sought $b_m$ for any positive integer $m$. We would simply start by introducing the factor $\sin(mx)$. Moreover, using the same orthogonality property, we could pick on the $a$’s by starting with the factor $\cos(mx)$—including the constant term since $\cos(0 \cdot x) = 1$. The only minor difference we want to be aware of is that

$$\int_{-\pi}^{\pi} \cos^2(mx) \, dx = \begin{cases} 2\pi, & m = 0 \\ \pi, & m \geq 1 \end{cases}$$

Careful consideration of this sheds light on why it is conventional to take the constant to be $\frac{a_0}{2}$ as opposed to just $a_0$. 
The Fourier Series of $f(x)$ on $(-\pi, \pi)$

The **Fourier series** of the function $f$ defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx,$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$
Some Useful Observations

Integer multiples of $\pi$:

- For every integer $n$, $\sin(n\pi) = 0$,

- and $\cos(n\pi) = 1$ if $n$ is even and $\cos(n\pi) = -1$ if $n$ is odd. Thus we write

$$\cos(n\pi) = (-1)^n.$$ 

Symmetry:

- The sine function is odd, $\sin(-\theta) = -\sin(\theta)$,

- and the cosine function is even, $\cos(-\theta) = \cos(\theta)$. 
Example

Find the Fourier series of the piecewise defined function

\[ f(x) = \begin{cases} 
0, & -\pi < x < 0 \\
x, & 0 \leq x < \pi 
\end{cases} \]

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos(nx) + b_n \sin(nx) \right) \]

\[ a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{-\pi}^{0} 0 \, dx + \frac{1}{\pi} \int_{0}^{\pi} x \, dx \]

\[ = \frac{1}{\pi} \left[ \frac{x^2}{2} \right]_{0}^{\pi} = \frac{1}{\pi} \left( \frac{\pi^2}{2} - \frac{0^2}{2} \right) = \frac{\pi^2}{2} \pi = \frac{\pi^2}{2} \]
\[ a_0 = \frac{\pi}{2} \]

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx = \frac{1}{\pi} \int_{-\pi}^{0} \cos(nx) \, dx + \frac{1}{\pi} \int_{0}^{\pi} x \cos(nx) \, dx \]

\[ = \frac{1}{\pi} \int_{0}^{\pi} x \cos(nx) \, dx \]

**Parts**

\( u = x \quad du = dx \)

\( v = \frac{1}{n} \sin(nx) \quad dv = \cos(nx) \, dx \)

\[ = \frac{1}{\pi} \left[ \frac{x}{n} \sin(nx) \right]_{0}^{\pi} - \frac{1}{n} \int_{0}^{\pi} \sin(nx) \, dx \]

\[ = \frac{1}{\pi} \left[ \frac{\pi}{n} \sin(n\pi) - \frac{0}{n} \sin(0) + \frac{1}{n^2} \cos(nx) \right]_{0}^{\pi} \]

\[ = \frac{1}{\pi} \left[ \frac{\pi}{n} \sin(n\pi) + \frac{1}{n^2} \cos(n\pi) \right] \]
\[ a_n = \frac{1}{\pi n^2} \left( (-1)^n - 1 \right) \]

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx = \frac{1}{\pi} \int_{0}^{\pi} 0 \cdot \sin(nx) \, dx + \frac{1}{\pi} \int_{0}^{\pi} x \sin(nx) \, dx \]

\[ = \frac{1}{\pi} \int_{0}^{\pi} x \sin(nx) \, dx \]

\[ \text{Parts} \]

\[ u = x \quad du = dx \]

\[ v = -\frac{1}{n} \cos(nx) \quad dv = \sin(nx) \, dx \]
\[
\begin{align*}
&= \frac{1}{\pi} \left[ -\frac{X}{n} \operatorname{Cor}(nx) \right]_0^\pi + \frac{1}{n} \int_0^\pi \operatorname{Cor}(nx) \, dx \\
&= \frac{1}{\pi} \left[ \frac{-\pi}{n} \operatorname{Cor}(n\pi) - \frac{0}{n} \operatorname{Cor}(0) + \frac{1}{n^2} \sin(nx) \right]_0^\pi \\
&= \frac{1}{\pi} \left[ \frac{-\pi}{n} (-1)^n + \frac{1}{n^2} \sin(n\pi) - \frac{1}{n^2} \sin(0) \right] \\
&= \frac{1}{\pi} \left( \frac{-\pi}{n} \right) (-1)^n = -\frac{(-1)^n}{n} = \frac{(-1)^{n+1}}{n}
\end{align*}
\]

\[b_n = \frac{(-1)^{n+1}}{n}\]
\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{8} a_n \cos(nx) + b_n \sin(nx) \]

\[ = \frac{\pi}{4} + \sum_{n=1}^{8} \left( \frac{1}{\pi n^2}((-1)^n-1) \cos(nx) + \frac{(-1)^n}{n} \sin(nx) \right) \]