November 9 Math 3260 sec. 57 Fall 2017

Section 5.2: The Characteristic Equation

Definition: For $n \times n$ matrix A, the expression

$$\det(A - \lambda I)$$

is an n^{th} degree polynomial in λ. It is called the **characteristic polynomial** of A.

Definition: The equation

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of A.

Theorem: The scalar λ is an eigenvalue of the matrix A if and only if it is a root of the characteristic equation.

Multiplicities

There are two types of *multiplicity* that can be associated with an eigenvalue λ of any given matrix.

Definition: The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation. The **geometric multiplicity** is the dimension of its corresponding eigenspace.

Example

Find all of the eigenvalues of the matrix A. Determine the algebraic and geometric multiplicities of each eigenvalue.

$$A = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 7-\lambda & 0 & -3 \\ -9 & -2-\lambda & 3 \\ 18 & 0 & -8-\lambda \end{bmatrix}\right)$$

Cofactor expansion down column 2

$$= \overset{=0}{a_{12}} C_{12} + a_{22} C_{22} + \overset{=0}{a_{32}} C_{32}$$

$$= (-2-\lambda) \begin{vmatrix} 7-\lambda & -3 \\ 18 & -8-\lambda \end{vmatrix}$$

$$= (-2-\lambda)\Big((7-\lambda)(-8-\lambda)+54\Big)$$

$$= (-2-\lambda)\left(-56+8\lambda-7\lambda+\lambda^2+54\right)$$

$$= (-2-\lambda)\left(\lambda^2+\lambda-2\right)$$

$$= -(2+\lambda)(\lambda+2)(\lambda-1)$$

$$= -(\lambda+2)^2(\lambda-1)$$

Char. eqn. $0 = -(\lambda+2)^2(\lambda-1)$ the eigenvalues are

$\lambda_1 = \lambda_2 = -2, \quad \lambda_3 = 1.$

$\lambda_1 = -2$ has algebraic multiplicity two.

$\lambda_3 = 1$ has algebraic multiplicity one.

We'll find bases for the eigenspaces.

For $\lambda_1 = \lambda_2 = -2$

$$(A - (-2)I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} 9 & 0 & -3 \\ -9 & 0 & 3 \\ 18 & 0 & -6 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = \frac{1}{3}x_3$

x_2, x_3 free

Eigenvectors look like

$$\vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}$$

A basis is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

The geometric multiplicity for $\lambda_1 = \lambda_2 = -2$ is two.

For $\lambda_3 = 1$

$$A - 1I = \begin{bmatrix} 6 & 0 & -3 \\ -9 & -3 & 3 \\ 18 & 0 & -9 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \frac{1}{2}x_3 \qquad x_3 \text{ - free}$$
$$x_2 = \frac{-1}{2}x_3$$

An eigenvector looks like

$$\vec{x} = x_3 \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

a basis is $\left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$.

The geometric multiplicity for $\lambda_3 = 1$ is one.

Similarity

Definition: Two $n \times n$ matrices A and B are said to be **similar** if there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

The mapping $A \mapsto P^{-1}AP$ is called a **similarity transformation**[1].

Theorem: If A and B are similar matrices, then they have the same characteristic equation, and hence the same eigenvalues.

[1] **Note that similarity is NOT related to being row equivalent.**

If $B = P^{-1}AP$, then $\det(B - \lambda I) = \det(A - \lambda I)$

Note that $I = P^{-1}P = P^{-1}IP$

$$\det(B-\lambda I) = \det(P^{-1}AP - \lambda I)$$

$$= \det(P^{-1}AP - \lambda P^{-1}IP)$$

$$= \det(P^{-1}(AP - \lambda IP))$$

$$= \det(P^{-1}(A - \lambda I)P)$$

$$= \det(P^{-1}) \det(A - \lambda I) \det(P)$$

scaler mult. commutes

$$= \underbrace{\det(P^{-1})\det(P)}_{1}\det(A - \lambda I)$$

$$= \det(A - \lambda I)$$

Example

Show that $A = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$ are similar with the matrix P for the similarity transformation given by $P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$.

$$P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 6 & -12 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix} = B$$

Example Continued...

Show that the columns of P are eigenvectors of A where

$$A = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}.$$

$$A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} \leftarrow \text{eigenvector for } \lambda_1 = 3$$

$$A\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ -4 \end{bmatrix} = -4\begin{bmatrix} 3 \\ 1 \end{bmatrix} \leftarrow \text{eigenvector for } \lambda_2 = -4$$

Eigenvalues of a real matrix need not be real

Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 3 \\ -5 & 2 \end{bmatrix}$.

Characteristic eqn

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 4-\lambda & 3 \\ -5 & 2-\lambda \end{bmatrix}\right) = (4-\lambda)(2-\lambda)+15$$

$$= \lambda^2 - 6\lambda + 8 + 15 = \lambda^2 - 6\lambda + 23$$

$$0 = \lambda^2 - 6\lambda + 23 = \lambda^2 - 6\lambda + 9 + 14$$

$$0 = (\lambda-3)^2 + 14$$

The roots $\quad -14 = (\lambda-3)^2 \Rightarrow \lambda - 3 = \pm\sqrt{-14}$

$$\lambda = 3 \pm \sqrt{14}\, i$$