November 9 Math 3260 sec. 58 Fall 2017

Section 5.2: The Characteristic Equation

Definition: For $n \times n$ matrix A, the expression

$$\det(A - \lambda I)$$

is an n^{th} degree polynomial in λ. It is called the **characteristic polynomial** of A.

Definition: The equation

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of A.

Theorem: The scalar λ is an eigenvalue of the matrix A if and only if it is a root of the characteristic equation.

Multiplicities

There are two types of *multiplicity* that can be associated with an eigenvalue λ of any given matrix.

Definition: The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation. The **geometric multiplicity** is the dimension of its corresponding eigenspace.

Example

Find all of the eigenvalues of the matrix A. Determine the algebraic and geometric multiplicities of each eigenvalue.

$$A = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}$$

Charact. Polynomial

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 7-\lambda & 0 & -3 \\ -9 & -2-\lambda & 3 \\ 18 & 0 & -8-\lambda \end{bmatrix}\right)$$

Cofactor expansion down column two

$$= \overset{=0}{a_{12}} C_{12} + a_{22} C_{22} + \overset{=0}{a_{32}} C_{32}$$

$$= (-2-\lambda)\begin{vmatrix} 7-\lambda & -3 \\ 18 & -8-\lambda \end{vmatrix}$$

$$= (-2-\lambda)\Big((7-\lambda)(-8-\lambda)+54\Big)$$

$$= (-2-\lambda)\left(-56+8\lambda-7\lambda+\lambda^2+54\right)$$

$$= -(2+\lambda)\left(\lambda^2+\lambda-2\right)$$

$$= -(2+\lambda)(\lambda+2)(\lambda-1) = -(\lambda+2)^2(\lambda-1)$$

The characteristic eqn is

$0 = -(\lambda+2)^2(\lambda-1)$ with roots $\lambda_1 = \lambda_2 = -2$, $\lambda_3 = 1$.

The algebraic multiplicity of -2 is two.

The algebraic multiplicity of 1 is one.

We can finded bases for the eigen spaces.

$$A-(-2)I = \begin{bmatrix} 9 & 0 & -3 \\ -9 & 0 & 3 \\ 18 & 0 & -6 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = \frac{1}{3}x_3$

x_2, x_3 – free

Eigenvectors look like

$$\vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}$$

a basis is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$

The geometric multiplicity of -2 is two.

For $\lambda_3 = 1$

$$A - I = \begin{bmatrix} 6 & 0 & -3 \\ -9 & -3 & 3 \\ 18 & 0 & -9 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = \frac{1}{2} x_3$

$x_2 = \frac{-1}{2} x_3$

x_3 - free

The eigenvectors have the form

$$\vec{x} = x_3 \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

a basis for the eigenspace is $\left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$

The geometric multiplicity of $\underline{1}$ is one.

Similarity

Definition: Two $n \times n$ matrices A and B are said to be **similar** if there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

The mapping $A \mapsto P^{-1}AP$ is called a **similarity transformation**[1].

Theorem: If A and B are similar matrices, then they have the same characteristic equation, and hence the same eigenvalues.

[1] **Note that similarity is NOT related to being row equivalent.**

If $B = P^{-1}AP$, then $\det(B - \lambda I) = \det(A - \lambda I)$

Note $I = P^{-1} I P$.

$$\det(B - \lambda I) = \det(P^{-1}AP - \lambda I)$$

$$= \det(P^{-1}AP - \lambda P^{-1} I P)$$

$$= \det(P^{-1}(AP - \lambda I P))$$

$$= \det(P^{-1}(A - \lambda I)P)$$

$$= \det(P^{-1}) \det(A - \lambda I) \det(P)$$

scaler multiplication commutes

$$= \det(A-\lambda I)\underbrace{\det(P^{-1})\det(P)}_{=1}$$

$$= \det(A-\lambda I)$$

Example

Show that $A = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$ are similar with the matrix P for the similarity transformation given by $P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$.

$$P^{-1} = \frac{1}{\det P}\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = -\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 6 & -12 \\ 3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix} = B$$

Example Continued...

Show that the columns of P are eigenvectors of A where

$$A = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}.$$

$$A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$ ← eigenvector w/ eigenvalue 3

$$A\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ -4 \end{bmatrix} = -4\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$ ← eigenvector w/ eigenvalue -4

Eigenvalues of a real matrix need not be real

Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 3 \\ -5 & 2 \end{bmatrix}$.

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 4-\lambda & 3 \\ -5 & 2-\lambda \end{bmatrix}\right) = (4-\lambda)(2-\lambda)+15$$

$$= \lambda^2 - 6\lambda + 8 + 15 = \lambda^2 - 6\lambda + 23$$

$$0 = \lambda^2 - 6\lambda + 23 = \lambda^2 - 6\lambda + 9 + 14$$

$$= (\lambda-3)^2 + 14$$

$$\Rightarrow (\lambda-3)^2 = -14 \implies \lambda - 3 = \pm\sqrt{-14} = \pm\sqrt{14}\, i$$

The eigenvalues are $\lambda_{1,2} = 3 \pm \sqrt{14}\, i$