October 10 MATH 1113 sec. 52 Fall 2018

Section 5.3: Inverse of an Exponential Function

Definition: Let $a > 0$ and $a \neq 1$. For $x > 0$ define $\log_a(x)$ as a number such that

$$\text{if} \quad y = \log_a(x) \quad \text{then} \quad x = a^y.$$

The function

$$F(x) = \log_a(x)$$

is called the **logarithm function of base** a. It has domain $(0, \infty)$, range $(-\infty, \infty)$, and if $f(x) = a^x$ then

$$F(x) = f^{-1}(x).$$

In particular

- $\log_a(a^x) = x$ for every real x, and
- $a^{\log_a(x)} = x$ for every $x > 0$.

Evaluating Logs with a Calculator

Most logarithms can't be evaluated by hand. For example, $\log_e(70) \approx 4.2485$. No one memorizes that! Calculators tend to have two built in logarithm functions.

- **Common Log** base 10 denoted as log (note there is no subscript), and $\log(x) = \log_{10}(x)$
- **Natural Log** base e denoted[1] ln $\ln(x) = \log_e(x)$

In Calculus, you'll find that the prefered base is e—the natural log. Base 10 logs still have some use such as in the Richter scale used to measure earthquakes.

[1] The order LN instead of NL is probably due to the French name *le Logarithme Naturel* for this log.

Evaluating Logs with a Calculator

Suppose we wish to evaluate $\log_2(15)$. You turn to a calculator, but there is no $\log_2$ key! Fortunately, you're not out of luck. Every log can be stated in term of any other log by the following theorem:

Theorem: (Change of Base) Let a, b, and M be any positive numbers, then

$$\log_b(M) = \frac{\log_a(M)}{\log_a(b)}.$$

What this says is that you can turn your $\log_2$ problem into a log or ln problem, and use your machine!

Change of Base

Here's the meat of our theorem again: $\log_b(M) = \dfrac{\log_a(M)}{\log_a(b)}$.

Express $\log_2(15)$ in terms of the natural log. Use a calculator to approximate its value.

$$\log_2(15) = \frac{\ln(15)}{\ln(2)}$$

$$\approx 3.9069$$

$b = 2 \quad M = 15$

$a = e$

$\log_e(\ast) = \ln(\ast)$

Question

Here's the meat of our theorem again: $\log_b(M) = \dfrac{\log_a(M)}{\log_a(b)}$.

Suppose you wish to evaluate $\log_\pi(42)$ using a calculator. Which of the following expressions could be used to find the desired value?

(a) $\dfrac{\log(42)}{\log(\pi)}$

(b) $\ln\left(\dfrac{42}{\pi}\right)$

(c) $\dfrac{\ln(\pi)}{\ln(42)}$

(d) $\ln\left(\dfrac{\pi}{42}\right)$

$\log_\pi(42) = \dfrac{\log_a(42)}{\log_a(\pi)}$

$* \ \ln\left(\dfrac{x}{y}\right) \neq \dfrac{\ln(x)}{\ln(y)}$

Section 5.4: Properties of Logarithms

Let's recall the basic properties of exponentials. The logarithms have analog properties.

Let $a > 0$, with $a \neq 1$. Then for any real x and y

- $a^{x+y} = a^x \cdot a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$

Log of a Product

Theorem: Let M and N be any positive numbers and $a > 0$ with $a \neq 1$. Then

$$\log_a(MN) = \log_a(M) + \log_a(N).$$

Illustrative Example:

$$\log_2(16) = \log_2(2 \cdot 8) = \log_2(2) + \log_2(8).$$

Note that this equation is the true statement

$$4 = 1 + 3.$$

Question

Here's the meat of our theorem: $\log_a(MN) = \log_a(M) + \log_a(N)$.

Which of the following is equivalent to $\log_3(15)$?

(a) $\log_3(5) + \log_3(3)$

$= \log_3(5 \cdot 3)$

$= \log_3(5) + \log_3(3)$

(b) $\log_3(10) + \log_3(5)$

(c) $\log_3(3) \cdot \log_3(5)$

(d) all of the above are equivalent

(e) none of the above is equivalent

Log of a Power

Theorem: Let M be any positive number, a be any positive number different from 1, and p be any real number. Then

$$\log_a(M^p) = p\log_a(M).$$

Illustrative Example:

$$\log_2(64) = \log_2(4^3) = 3\log_2(4).$$

Note that this equation is the true statement

$$6 = 3 \cdot 2.$$

Question

Here's the meat of our theorem: $\log_a(M^p) = p \log_a(M)$.

Which of the following is equivalent to $\log_7(125)$?

(a) $5 \log_7(25)$

(b) $3 \log_7(5)$

(c) $(\log_7(5))^3$

(d) all of the above are equivalent

(e) none of the above is equivalent

$= \log_7(5^3)$

$= 3\log_7(5)$

Log of a Quotient

Theorem: Let M and N be any positive numbers and $a > 0$ with $a \neq 1$. Then

$$\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N).$$

Illustrative Example:

$$\log_2(4) = \log_2\left(\frac{16}{4}\right) = \log_2(16) - \log_2(4).$$

Note that this equation is the true statement

$$2 = 4 - 2.$$

Question

Here's the meat of our theorem: $\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$.

Which of the following is equivalent to $\log_3(5)$?

(a) $\log_3(15) - \log_3(3)$

(b) $\log_3(10) - \log_3(5)$

(c) $\frac{\log_3(15)}{\log_3(3)}$

(d) all of the above are equivalent

(e) none of the above is equivalent

$5 = \frac{15}{3}$

so

$\log_3(5) = \log_3\left(\frac{15}{3}\right)$

$= \log_3(15) - \log_3(3)$

$\Rightarrow \log_3(15) = \log_3(5) + \log_3(3)$