Oct 10 Math 2306 sec. 53 Fall 2018

Section 10: Variation of Parameters

We are still considering nonhomogeneous, linear ODEs. Consider equations of the form

$$y'' + y = \tan x, \quad \text{or} \quad x^2y'' + xy' - 4y = e^x.$$

The method of undetermined coefficients is not applicable to either of these. We require another approach.

Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

$y_c = c_1 y_1(x) + c_2 y_2(x)$

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

We have one equation (the ODE) and 2 unknowns u_1, u_2. We'll introduce a 2^{nd} equation in the process.

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2' + \underbrace{u_1' y_1 + u_2' y_2}_{\text{we'll set this to zero}}$$

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

* 2^{nd} Condition $\quad u_1' y_1 + u_2' y_2 = 0$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Sub into $\quad y_p'' + P(x) y_p' + Q(x) y_p = g(x)$

$$u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' + P(x)\left(u_1 y_1' + u_2 y_2'\right) + Q(x)\left(u_1 y_1 + u_2 y_2\right) = g(x)$$

Collect by u_1 and u_2

$$u_1' y_1' + u_2' y_2' + u_1 \underbrace{\left(y_1'' + P(x) y_1' + Q(x) y_1\right)}_{= 0} + u_2 \underbrace{\left(y_2'' + P(x) y_2' + Q(x) y_2\right)}_{= 0} = g(x)$$

The ODE reduces to

$$u_1' y_1' + u_2' y_2' = g(x)$$

We have to solve the system of equations

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = g(x)$$

In matrix format, this is

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

note that's the Wronskian matrix!

Using Crammer's rule

$$u_1' = \frac{W_1}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W}$$

where $W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}$, $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$u_1' = \frac{0 - g y_2}{W} \quad \text{and} \quad u_2' = \frac{y_1 g - 0}{W}$$

$$u_1 = \int \frac{-g(x) y_2(x)}{W} \, dx \qquad u_2 = \int \frac{y_1(x) g(x)}{W} \, dx$$

And $\quad y_p = u_1 y_1 + u_2 y_2$

Example:

Solve the ODE $y'' + y = \tan x$. $\leftarrow$ Standard form

Find y_c : $m^2+1=0 \Rightarrow m = \pm i$

$y_1(x) = \cos(x) \quad y_2(x) = \sin(x)$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$g(x) = \tan x$

$$u_1 = \int \frac{-g y_2}{W}\,dx = \int \frac{-\tan x \sin x}{1}\,dx$$

$$= \int (\cos x - \sec x)\,dx = \sin x - \ln|\sec x + \tan x|$$

$$u_2 = \int \frac{g\, y_1}{W}\,dx = \int \frac{\tan x \cos x}{1}\,dx$$

$$= \int \sin x\,dx = -\cos x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(\sin x - \ln|\sec x + \tan x|\right)\cos x + (-\cos x)\sin x$$

$$= -\cos x \ln|\sec x + \tan x|$$

The general solution is

$$y = C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x + \tan x|$$