October 10 Math 2306 sec. 56 Fall 2017

Section 10: Variation of Parameters

We are still considering nonhomogeneous, linear ODEs. Consider equations of the form

$$y'' + y = \tan x, \quad \text{or} \quad x^2y'' + xy' - 4y = e^x.$$

The method of undetermined coefficients is not applicable to either of these. We require another approach.

Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x) = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

We have one equation and two unknowns. We'll introduce a second condition that is convenient.

$$y_p' = u_1y_1' + u_2y_2' + \underbrace{u_1'y_1 + u_2'y_2}_{\text{"}0\text{"}}$$

$$y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''$$

We'll assume

$$u_1'y_1 + u_2'y_2 = 0$$

Substitute $\quad y_p'' + P(x)y_p' + Q(x)y_p = g(x)$

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

$$u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'' + P(x)(u_1y_1' + u_2y_2') + Q(x)(u_1y_1 + u_2y_2) = g(x)$$

Collect by u_1 and u_2

$$u_1'y_1' + u_2'y_2' + u_1\underbrace{\left(y_1'' + P(x)y_1' + Q(x)y_1\right)}_{=0} + u_2\underbrace{\left(y_2'' + P(x)y_2' + Q(x)y_2\right)}_{=0} = g(x)$$

Since y_1, y_2 solve the homogeneous eqn.

Our second equation is

$$u_1'y_1' + u_2'y_2' = g(x)$$

To find u_1 and u_2, we need to solve

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = g(x)$$

In a matrix format, this is

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}$$

Crammer's Rule: the system

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$ has solution

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}$$ where

$$D_1 = \det\begin{pmatrix} e & b \\ f & d \end{pmatrix}, \quad D_2 = \det\begin{pmatrix} a & e \\ c & f \end{pmatrix}$$ and

$$D = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ provided $D \neq 0$.

Using Crammer's Rule

$$u_1' = \frac{W_1}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W}$$

where
$$W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}$$

and
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$ the wronskian of y_1 and y_2.

$$u_1' = \frac{-g y_2}{W} \quad \text{and} \quad u_2' = \frac{g y_1}{W}$$

$$u_1 = \int \frac{-g y_2}{W}\, dx$$

$$u_2 = \int \frac{g y_1}{W}\, dx$$

$$y_p = u_1 y_1 + u_2 y_2$$

Example:

Solve the ODE $y'' + y = \tan x$.

It's in standard form with
$g(x) = \tan x$

Solve the associated homogeneous:

$y'' + y = 0 \implies m^2 + 1 = 0$

$m = \pm i \qquad \alpha = 0, \ \beta = 1$

$y_1 = \cos x \ , \ y_2 = \sin x$

We have to keep this order from here on out.

Wronskian

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$u_1 = \int -\frac{g y_2}{W}\,dx \qquad u_2 = \int \frac{g y_1}{W}\,dx$$

$$u_1 = \int \frac{-\tan x \sin x}{1}\,dx = -\int \tan x \sin x\,dx$$

$$= -\int \frac{\sin x}{\cos x}\sin x\,dx = -\int \frac{\sin^2 x}{\cos x}\,dx$$

$$= -\int \frac{1-\cos^2 x}{\cos x}\,dx = -\int \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}\,dx$$

$$= -\int (\sec x - \cos x)\,dx$$

$$= -\left(\ln|\sec x + \tan x| - \sin x\right)$$

$$u_1 = -\ln\left|\sec x + \tan x\right| + \sin x$$

$$u_2 = \int \frac{\tan x \cos x}{1}\,dx = \int \frac{\sin x}{\cos x}\cos x\,dx$$

$$= \int \sin x\,dx = -\cos x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \cos x\left(-\ln\left|\sec x + \tan x\right| + \sin x\right) + (-\cos x)\sin x$$

$$= -\cos x \ln\left|\sec x + \tan x\right| + \cos x \sin x - \cos x \sin x$$

$$y_p = -\cos x \ln|\sec x + \tan x|$$

$\boxed{\text{The general solution to the nonhomogeneous ODE is } y = c_1\cos x + c_2\sin x - \cos x\ln|\sec x + \tan x|}$

Example:

Solve the ODE

$$x^2y'' + xy' - 4y = \ln x,$$

given that $y_c = c_1x^2 + c_2x^{-2}$ is the complementary solution.

Standard form: $y'' + \frac{1}{x}y' - \frac{4}{x^2}y = \frac{\ln x}{x^2}$

$g(x) = \frac{\ln x}{x^2}$, $y_1 = x^2$, $y_2 = x^{-2}$

$$W = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = -2x^{-1} - 2x^{-1} = -4x^{-1}$$

$u_1 = \int \frac{-gy_2}{W}\,dx$ $\qquad$ $u_2 = \int \frac{gy_1}{W}\,dx$

$$u_1 = \int -\frac{\frac{\ln x}{x^2} \cdot x^{-2}}{-4x^{-1}}\, dx = \frac{1}{4}\int x^{-3} \ln x \, dx$$

Int by parts

$$u = \ln x \qquad du = \frac{1}{x}\, dx$$

$$v = \frac{-x^{-2}}{2} \qquad dv = x^{-3}\, dx$$

$$u_1 = \frac{1}{4}\left(\frac{-1}{2} x^{-2} \ln x - \int \frac{-1}{2} x^{-2} \cdot x^{-1}\, dx \right)$$

$$= \frac{1}{4}\left(\frac{-1}{2} x^{-2} \ln x + \frac{1}{2}\int x^{-3}\, dx \right)$$

$$= \frac{1}{4}\left(\frac{-1}{2} x^{-2} \ln x - \frac{1}{4} x^{-2} \right)$$

$$u_2 = \int \frac{g\, y_1}{W}\, dx = \int \frac{\frac{\ln x}{x^2} \cdot x^2}{-4x^{-1}}\, dx$$

$$= \frac{-1}{4} \int x \ln x \, dx$$

Int by parts $\quad u = \ln x \quad du = \frac{1}{x}\, dx$

$\quad v = \frac{x^2}{2} \quad dv = x\, dx$

$$= \frac{-1}{4}\left(\frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\, dx \right)$$

$$= \frac{-1}{4}\left(\frac{x^2}{2} \ln x - \frac{1}{2} \int x\, dx \right) = \frac{-1}{4}\left(\frac{x^2}{2} \ln x - \frac{x^2}{4} \right)$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{1}{4}\left(\frac{-1}{2}\ln x - \frac{1}{4}\right)x^{-2}\cdot x^2 - \frac{1}{4}\left(\frac{1}{2}\ln x - \frac{1}{4}\right)x^2\cdot x^{-2}$$

$$= \frac{-1}{8}\ln x - \frac{1}{16} - \frac{1}{8}\ln x + \frac{1}{16}$$

$$= \frac{-1}{4}\ln x$$

The general solution to the nonhomogeneous ODE is

$$\boxed{y = c_1 x^2 + c_2 x^{-2} - \frac{1}{4}\ln x}$$