October 10 Math 2306 sec. 57 Fall 2017

Section 10: Variation of Parameters

We are still considering nonhomogeneous, linear ODEs. Consider equations of the form

$$y''+y=\tan x, \quad \text{or} \quad x^2y''+xy'-4y=e^x.$$

The method of undetermined coefficients is not applicable to either of these. We require another approach.

Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x) = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

We have 2 unknowns and one equation. So we'll impose a second equation.

$$y_p' = u_1y_1' + u_2y_2' + \underbrace{u_1'y_1 + u_2'y_2}_{\overset{\shortparallel}{0}}$$

We'll impose

$$u_1'y_1 + u_2'y_2 = 0$$

$$y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''$$

Substitute $y_p'' + P(x)y_p' + Q(x)y_p = g(x)$

Remember that $y_i'' + P(x)y_i' + Q(x)y_i = 0$, for $i = 1, 2$

$$u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'' + P(x)\left(u_1y_1' + u_2y_2'\right) + Q(x)\left(u_1y_1 + u_2y_2\right) = g(x)$$

Collect by u_1 and u_2

$$u_1'y_1' + u_2'y_2' + u_1\underbrace{\left(y_1'' + P(x)y_1' + Q(x)y_1\right)}_{\overset{\shortparallel}{0}} + u_2\underbrace{\left(y_2'' + P(x)y_2' + Q(x)y_2\right)}_{\overset{\shortparallel}{0}} = g(x)$$

Since y_1 and y_2 solve the homogeneous equation

So our second equation is

$$u_1'y_1' + u_2'y_2' = g(x)$$

So the u's solve the system

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = g(x)$$

In matrix format, this is

$$\underbrace{\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}}_{\text{this is the matrix for the wronskian of } y_1 \text{ and } y_2} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

Crammer's Rule :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$

$x_1 = \dfrac{D_1}{D}$, $x_2 = \dfrac{D_2}{D}$ where

$$D_1 = \begin{vmatrix} e & b \\ f & d \end{vmatrix} \qquad D_2 = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$$

and $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ provided $D \neq 0$.

Using Crammer's rule

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

$$u_1' = \frac{W_1}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W}$$

where $W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}$, $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}$

and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ the Wronskian of y_1, y_2.

$$u_1 = \int \frac{-g y_2}{W}\, dx \quad , \quad u_2 = \int \frac{g y_1}{W}\, dx$$

$$y_p = u_1 y_1 + u_2 y_2$$

Gen. Soln. $y = y_c + y_p$

Example:

Solve the ODE $y'' + y = \tan x$.

This is in standard form.

$g(x) = \tan x$

Assoc. Homogeneous:

$$y'' + y = 0$$

$$m^2 + 1 = 0 \Rightarrow m = \pm i \qquad \alpha = 0 \qquad \beta = 1$$

$$y_1 = \cos x \ , \quad y_2 = \sin x$$

Wronskian $$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$u_1 = \int \frac{-g y_2}{W}\, dx \qquad u_2 = \int \frac{g y_1}{W}\, dx$$

$$u_1 = \int -\frac{\tan x \sin x}{1}\, dx = \int -\tan x \sin x \, dx$$

$$= -\int \frac{\sin x}{\cos x} \sin x \, dx = -\int \frac{\sin^2 x}{\cos x}\, dx$$

$$= -\int \frac{1-\cos^2 x}{\cos x}\, dx = -\int \left(\frac{1}{\cos x} - \frac{\cos^2 x}{\cos x} \right) dx$$

$$= -\int (\sec x - \cos x)\, dx$$

$$= -\left(\ln|\sec x + \tan x| - \sin x \right)$$

$$u_1 = -\ln|\sec x + \tan x| + \sin x$$

$$u_2 = \int \frac{\tan x \cos x}{1}\,dx = \int \frac{\sin x}{\cos x} \cos x \, dx$$

$$= \int \sin x \, dx = -\cos x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p = (-\ln|\sec x + \tan x| + \sin x)\cos x + (-\cos x)\sin x$$

$$y_p = -\cos x \ln|\sec x + \tan x| + \sin x \cancel{\cos x} - \cancel{\cos x} \sin x$$

$$y_p = -\cos x \ln|\sec x + \tan x|$$

The general solution to the nonhomogeneous ODE is

$$y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x| .$$

Example:

Solve the ODE

$$x^2y'' + xy' - 4y = \ln x,$$

given that $y_c = c_1x^2 + c_2x^{-2}$ is the complementary solution.

Standard form: $y'' + \frac{1}{x}y' - \frac{4}{x^2}y = \frac{\ln x}{x^2}$

$g(x) = \frac{\ln x}{x^2}$, $y_1 = x^2$, $y_2 = x^{-2}$

Wronskian $W = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = -2x^{-1} - 2x^{-1} = -4x^{-1}$

$u_1 = \int \frac{-gy_2}{W}\,dx$ $\qquad u_2 = \int \frac{gy_1}{W}\,dx$

$$u_1 = \int -\frac{\left(\frac{\ln x}{x^2}\right) x^{-2}}{-4x^{-1}}\,dx = \frac{1}{4}\int x^{-3}\ln x\,dx$$

Int. by parts:

$u = \ln x \qquad du = \frac{1}{x}\,dx$

$v = \frac{-1}{2}x^{-2} \qquad dv = x^{-3}\,dx$

$$u_1 = \frac{1}{4}\left(\frac{-1}{2}x^{-2}\ln x - \int \frac{-1}{2}x^{-2}\cdot\frac{1}{x}\,dx\right)$$

$$= \frac{1}{4}\left(\frac{-1}{2}x^{-2}\ln x + \frac{1}{2}\int x^{-3}\,dx\right)$$

$$= \frac{1}{4}\left(\frac{-1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2}\right) = \frac{-x^{-2}}{16}\left(2\ln x + 1\right)$$

$$u_2 = \int \frac{\left(\frac{\ln x}{x^2}\right) \cdot x^2}{-4x^{-1}}\,dx = \frac{-1}{4}\int x \ln x \, dx$$

Int by parts

$u = \ln x \qquad du = \frac{1}{x}\,dx$

$v = \frac{x^2}{2} \qquad dv = x\,dx$

$$u_2 = \frac{-1}{4}\left(\frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx\right)$$

$$= \frac{-1}{4}\left(\frac{x^2}{2}\ln x - \frac{1}{2}\int x \, dx\right)$$

$$= \frac{-1}{4}\left(\frac{x^2}{2}\ln x - \frac{1}{4}x^2\right) = \frac{-x^2}{16}\left(2\ln x - 1\right)$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p = \frac{-x^{-2}}{16}(2\ln x + 1)x^2 - \frac{x^2}{16}(2\ln x - 1)x^{-2}$$

$$= -\frac{1}{8}\ln x - \frac{1}{16} - \frac{1}{8}\ln x + \frac{1}{16}$$

$$= -\frac{1}{4}\ln x$$

The general solution is

$$y = c_1 x^2 + c_2 x^{-2} - \frac{1}{4}\ln x$$