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Section 4.4: Coordinate Systems

Given an ordered basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ for a vector space V, we noted that the coefficients in the linear combination $\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n$ are unique for a given $\mathbf{x}$. This allows for an unambiguous definition of coordinate vectors.

Definition: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an **ordered** basis of the vector space V. For each $\mathbf{x}$ in V we define the **coordinate vector of x relative to the basis** $\mathcal{B}$ to be the unique vector $(c_1, \ldots, c_n)$ in $\mathbb{R}^n$ where these entries are the weights $\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n$.

We'll use the notation $[\mathbf{x}]_{\mathcal{B}}$ to denote such a coorindate vector.

For Example

For the elementary basis $\mathcal{B} = \{1, t, t^2, t^3\}$ (in that order) in $\mathbb{P}_3$, a typical vector $\mathbf{p}(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$ would have coordinate vector in $\mathbb{R}^4$,

$$[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

Coordinates in $\mathbb{R}^n$

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis of $\mathbb{R}^n$. Then the change of coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the linear transformation defined by

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$$

where the matrix

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n].$$

Example

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_1, \mathbf{b}_3\}$ where $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{b}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

For $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{B}}$.

$$P_{\mathcal{B}} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{\mathcal{B}}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\vec{x}]_B = P_B^{-1}\vec{x} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 \end{bmatrix}$$

Theorem: Coordinate Mapping

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a **one to one** mapping of V **onto** $\mathbb{R}^n$.

Remark: When such a mapping exists, we say that V is **isomorphic** to $\mathbb{R}^n$. Properties of subsets of V, such as linear dependence, can be discerned from the coordinate vectors in $\mathbb{R}^n$.

Example

Use coordinate vectors to determine if the set $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ is linearly dependent or independent in $\mathbb{P}_2$.

$$\mathbf{p}(t) = 1 - 2t^2, \quad \mathbf{q}(t) = 3t + t^2, \quad \mathbf{r}(t) = 1 + t$$

We can use the elementary basis $\{1, t, t^2\} = \mathcal{B}$

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad [\vec{q}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \quad [\vec{r}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{p} = 1\cdot 1 + 0\cdot t + (-2)t^2, \quad \vec{q} = 0\cdot 1 + 3\cdot t + 1\cdot t^2, \quad \vec{r} = 1\cdot 1 + 1\cdot t + 0\cdot t^2$$

We can use a determinant.

Let $A = \left[[\vec{p}]_B \quad [\vec{q}]_B \quad [\vec{r}]_B \right]$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 1 \\ -2 & 1 & 0 \end{bmatrix} \qquad \det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\det(A) = 1 \cdot \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} + 0 \cdot \begin{vmatrix} & \\ & \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 3 \\ -2 & 1 \end{vmatrix}$$

$$= 1(-1) + 1(6) = 5 \quad \neq 0$$

The columns of A are linearly independent, hence $\{\vec{p}, \vec{q}, \vec{r}\}$ is linearly independent.

Example

Let H be the subset of $M^{2\times 2}$ of matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

(a) Show that H is a subspace of $M^{2\times 2}$ by finding a set S of vectors such that $H = \text{Span}(S)$.

Let $\vec{u}$ be in H. Then $\vec{u} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for some real a, b.

$$\vec{u} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

we can take

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}.$$

Example

(b) Demonstrate that the set S is a basis, or remove elements of S in order to obtain a basis $\mathcal{B}$ for H.

S is a basis.

Suppose $c_1\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + c_2\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}c_1 & -c_2\\ c_2 & c_1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$\Rightarrow c_1 = c_2 = 0$ Hence S is linearly independent.

(c) Consider the mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ for your basis. Determine n such that H is isomorphic to $\mathbb{R}^n$.

Each linear combination has 2 coefficients – since there are 2 basis vectors. So $n = 2$.

Section 4.5: Dimension of a Vector Space

Theorem: If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set of vectors in V containing *more than n vectors* is linearly dependent.

We've seen the example that if a set of vectors from $\mathbb{R}^n$ has more than n vectors, i.e. more vectors than entries in each one, the set is linearly dependent.

Dimension

Corollary: If vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then every basis of V consist of exactly n vectors.

Definition: If V is spanned by a finite set, then V is called **finite dimensional**. In this case, the dimension of V

$$\dim V \;=\; \text{the number of vectors in any basis of } V.$$

The dimension of the vector space $\{\mathbf{0}\}$ containing only the zero vector is defined to be zero—i.e.

$$\dim\{\mathbf{0}\} = 0.$$

If V is not spanned by a finite set[1], then V is said to be **infinite dimensional**.

[1] $C^0(\mathbb{R})$ is an example of an infinite dimensional vector space.

Examples

(a) Find $\dim(\mathbb{R}^n)$.

$\dim(\mathbb{R}^n) = n$

The elementary basis, for example, has n vectors $\vec{e}_1, \ldots, \vec{e}_n$ in it.

(b) Determine dim Col A where $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$.

A basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$. Contains 2 vectors

$\dim \operatorname{Col} A = 2$.

Some Geometry in $\mathbb{R}^3$

Give a geometric description of subspaces of $\mathbb{R}^3$ of dimension

(a) zero This is $\{\vec{0}\}$. The origin in $\mathbb{R}^3$ is the only such subspace.

(b) one Must look like $\text{Span}\{\vec{u}\}$ where $\vec{u} \neq \vec{0}$.
Any line through the origin is such a subspace.

(c) two Must look like $\text{Span}\{\vec{u}, \vec{v}\}$ with $\vec{u}, \vec{v}$ linearly independent,
These are planes through the origin.

(d) three The only one is $\mathbb{R}^3$.

Subspaces and Dimension

Theorem: Let H be a subspace of a finite dimensional vector space V. Then H is finite dimensional and

$$\dim H \leq \dim V.$$

Moreover, any linearly independent subset of H can be expanded if needed to form a basis for H.

Theorem: Let V be a vector space with dim $V = p$ where $p \geq 1$. Any linearly independent set in V containing exactly p vectors is a basis for V. Similarly, any spanning set consisting of exactly p vectors in V is necessarily a basis for V.

Column and Null Spaces

Theorem: Let A be an $m \times n$ matrix. Then

$$\dim \text{Nul}A \ = \ \text{the number of free variables in the equation } \ A\mathbf{x} = \mathbf{0},$$

and

$$\dim \text{Col}A \ = \ \text{the number of pivot positions in } \ A.$$

Example

Find the dimensions of the null and columns spaces of the matrix A.

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & 1 & -7 & -1 \\ 3 & 0 & 6 & 1 \end{bmatrix}$$

We can use the rref of A

$$\Big\downarrow \text{rref}$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There are 3 pivot columns.

$\dim \operatorname{Col} A = 3$.

The number of free variables in $A\vec{x} = \vec{0}$ is the number of non pivot columns. Here that's 1.

So $\dim \operatorname{Nul} A = 1$

Section 4.6: Rank

Definition: The **row space**, denoted Row A, of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

Example: Express the row space of A in term of a span

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

we can take the rows of A.

$$\text{Row } A = \text{Span}\left\{ \begin{bmatrix} -2 \\ -5 \\ 8 \\ 0 \\ -17 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -5 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 11 \\ -19 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -13 \\ 5 \\ -3 \end{bmatrix} \right\}$$

Theorem

If two matrices A and B are row equivalent, then their row spaces are the same.

In particular, if B is an echelon form of the matrix A, then the nonzero rows of B form a basis for Row B—and also for Row A since these are the same space.

Example

A matrix A along with its rref is shown.

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for Row A and state the dimension dim Row A.

Using the nonzero rows of rref A, a basis is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -5 \end{bmatrix} \right\}$$

$\dim \text{Row} A = 3$

Example continued ...

(b) Find a basis for Col A and state its dimension.

Pivot columns are 1, 2 and 4. A basis is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

$\dim \mathrm{Col}\, A = 3$

Example continued ...

(c) Find a basis for Nul A and state its dimension.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For $\vec{x}$ in Nul A

$$x_1 = -x_3 - x_5$$
$$x_2 = 2x_3 - 3x_5$$
$$x_4 = 5x_5$$

x_3, x_5 - free

$$\vec{x} = \begin{bmatrix} -x_3 - x_5 \\ 2x_3 - 3x_5 \\ x_3 \\ 5x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

A basis is $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$

$\dim \mathrm{Nul}\, A = 2.$