October 12 MATH 1113 sec. 52 Fall 2018

Section 5.4: Properties of Logarithms

Let's recall the basic properties of exponentials. The logarithms have analog properties.

Let $a > 0$, with $a \neq 1$. Then for any real x and y

- $a^{x+y} = a^x \cdot a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$

Log Properties

For M and N positive, and p any real number

$$\begin{aligned}
\log_a(MN) &= \log_a(M) + \log_a(N) \\
\log_a(M^p) &= p\log_a(M) \\
\log_a\left(\frac{M}{N}\right) &= \log_a(M) - \log_a(N)
\end{aligned}$$

These properties hold for any base a. We also have the change of base formula

$$\log_b(M) = \frac{\log_a(M)}{\log_a(b)}$$

Example

Write the following as a single logarithm expression.

$$2\ln(x) - \frac{1}{3}\ln(2y) + 3\ln(3)$$

$$= \ln(x^2) - \ln\left((2y)^{\frac{1}{3}}\right) + \ln\left(3^3\right)$$

$$= \ln(x^2) - \ln\left(\sqrt[3]{2y}\right) + \ln(27)$$

$$= \ln\left(\frac{x^2}{\sqrt[3]{2y}}\right) + \ln(27)$$

$$= \ln\left(27\frac{x^2}{\sqrt[3]{2y}}\right) = \ln\left(\frac{27x^2}{\sqrt[3]{2y}}\right)$$

Question

$$4\log(x) + \frac{1}{2}\log(z) \quad \text{is equivalent to}$$

$= \log(x^4) + \log(z^{1/2})$

$= \log(x^4) + \log(\sqrt{z})$

$= \log(x^4\sqrt{z})$

also $= \log\left(\sqrt{x^8 z}\right)$

(a) $\log\left(\frac{4xz}{2}\right)$

(b) $\log\left(\frac{x^4}{\sqrt{z}}\right)$

(c) $\log(x^4\sqrt{z})$

(d) all of the above are equivalent

(e) none of the above is equivalent

Example

Use the change of base formula to determine $\log_3(x)$ in terms of the logarithm of base $\frac{1}{3}$.

$$\log_{\frac{1}{3}}(x) = \frac{\log_3(x)}{\log_3\left(\frac{1}{3}\right)} = \frac{\log_3(x)}{-1} = -\log_3(x)$$

Note $\frac{1}{3} = 3^{-1}$ $\quad \log_3\left(\frac{1}{3}\right) = \log_3\left(3^{-1}\right) = -1\log_3(3)$

$$= -1 \cdot 1 = -1$$

Question

True/False If $\log_2(x) = y$, then $\log_2\left(\frac{1}{x}\right) = -y$.

(a) True and I'm confident.

(b) True, but I'm not confident.

(c) False, and I'm confident.

(d) False, but I'm not confident.

$$\log_2\left(\frac{1}{x}\right) = \log_2\left(x^{-1}\right)$$
$$= -1\log_2(x)$$
$$= -1\cdot y$$
$$= -y$$

Question

Properties of logarithms are sometimes confused with erroneous, somewhat look-alike expressions.
Which of the following is FALSE?

(a) $\ln(xy) = (\ln x)(\ln y)$

$\log_2(x) = \ln\left(x^{\frac{1}{\ln 2}}\right)$

(b) $\log_2(x) = \ln(x^2)$

(c) $\log_4(2+7) = \log_4(2) + \log_4(7)$

(d) $(\log(10))^5 = \log(10^5)$

(e) All of the above are false.

Example: Using the properties together

Express the following as a sum, difference, and multiple of logarithms.

$$\log_a \sqrt{\frac{a^3 b^5}{\sqrt[3]{a}\sqrt{b+1}}} = \log_a \left(\frac{a^3 b^5}{a^{1/3} (b+1)^{1/2}} \right)^{1/2}$$

$$\frac{a^3}{a^{1/3}} = a^{3-\frac{1}{3}} = a^{8/3}$$

$$= \frac{1}{2} \log_a \left(\frac{a^{8/3} b^5}{(b+1)^{1/2}} \right)$$

$$= \frac{1}{2} \left(\log_a \left(a^{8/3} b^5 \right) - \log_a (b+1)^{\frac{1}{2}} \right)$$

$$= \frac{1}{2} \left(\log_a \left(a^{8/3} \right) + \log_a \left(b^5 \right) - \log_a (b+1)^{1/2} \right)$$

$$= \frac{1}{2}\left(\frac{8}{3}\log_a(a) + 5\log_a(b) - \frac{1}{2}\log_a(b+1) \right)$$

$$= \frac{1}{2}\left(\frac{8}{3}\cdot 1 + 5\log_a(b) - \frac{1}{2}\log_a(b+1) \right)$$

$$= \frac{4}{3} + \frac{5}{2}\log_a(b) - \frac{1}{4}\log_a(b+1)$$

Summary of Log Properties

Assume each expression is well defined.

(i) Change of base: $\log_b(M) = \dfrac{\log_a(M)}{\log_a(b)}$

(ii) Log of Product: $\log_a(MN) = \log_a(M) + \log_a(N)$

(iii) Log of Power: $\log_a(M^p) = p\log_a(M)$

(iv) Log of Quotient: $\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$

(v) Inverse Function: $a^{\log_a(x)} = x$ and $\log_a(a^x) = x$

(vi) Special Values: $\log_a(1) = 0$, $\log_a(a) = 1$, and $\log_a(0)$ is never defined.