Oct. 12 Math 1190 sec. 51 Fall 2016

Section 4.1: Related Rates

Motivating Example: A spherical balloon is being filled with air. Suppose that we know that the radius is increasing in time at a constant rate of 2 mm/sec. Can we determine the rate at which the surface area of the balloon is increasing at the moment that the radius is 10 cm?

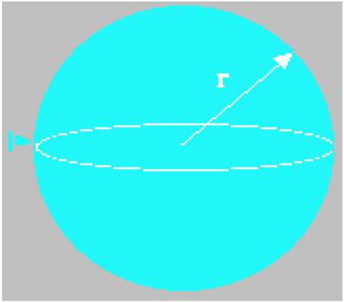


Figure: Spherical Balloon

Example Continued...

Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of time. Write an equation that relates

$$\frac{dS}{dt} \quad \text{to} \quad \frac{dr}{dt}.$$

$\frac{dS}{dt}$ - rate of change of surface area

$\frac{dr}{dt}$ - rate of change of radius.

From $S = 4\pi r^2$ and the chain rule

$$\frac{d}{dt} S = \frac{d}{dt}(4\pi r^2)$$

$$\frac{dS}{dt} = 4\pi(2r)\frac{dr}{dt} = 8\pi r \frac{dr}{dt}$$

Given this result, find the rate at which the surface area is changing when the radius is 10 cm.

we have $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

Since r is increasing @ 2mm/sec, $\frac{dr}{dt} = 2 \frac{mm}{sec}$

when $r = 10$ cm $= 100$ mm,

$$\frac{dS}{dt} = 8\pi\,(100\text{ mm})\left(2\,\frac{\text{mm}}{\text{sec}}\right)$$

$$= 1600\pi\,\frac{\text{mm}^2}{\text{sec}}$$

At this moment, the surface area is

increasing at a rate of $1600\pi \ \frac{mm^2}{sec}$.

Example

A right circular cone of height h and base radius r has volume

$$V = \frac{\pi}{3} r^2 h.$$

(a) Find $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$ if r is constant.

$$\frac{d}{dt} V = \frac{d}{dt}\left(\frac{\pi}{3} r^2 h\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3} r^2 \frac{dh}{dt}$$

since r, hence r^2, is constant.

Example Continued...

$$V = \frac{\pi}{3} r^2 h$$

Question (b) Find $\frac{dV}{dt}$ in terms of $\frac{dr}{dt}$ if h is constant.

(a) $\frac{dV}{dt} = \frac{2\pi}{3} r \frac{dr}{dt}$

(b) $\frac{dV}{dt} = \frac{\pi}{3} r^2 \frac{dr}{dt}$

(c) $\frac{dV}{dt} = \frac{\pi}{3} r^2 h \frac{dr}{dt}$

(d) $\frac{dV}{dt} = \frac{2\pi}{3} rh \frac{dr}{dt}$

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{\pi}{3} r^2 h\right)$$

$$= \frac{\pi}{3} h \, (2r) \frac{dr}{dt}$$

$$= \frac{2\pi}{3} rh \frac{dr}{dt}$$

And Continued Further...

$$V = \frac{\pi}{3} r^2 h$$

(c) Find $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$ and $\frac{dr}{dt}$ assuming neither r nor h is constant.

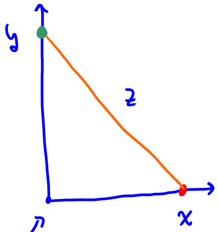
$$\frac{d}{dt} V = \frac{d}{dt}\left(\frac{\pi}{3} \underbrace{r^2 h}_{\text{product}}\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left(h\,(2r)\frac{dr}{dt} + r^2 \frac{dh}{dt}\right)$$

$$= \frac{2\pi}{3} r h \frac{dr}{dt} + \frac{\pi}{3} r^2 \frac{dh}{dt}$$

Example

Two cars leave from the same place. One heads due east at 30 mph, the other heads due north at a rate of 45 mph. At what rate is the distance between them changing after 15 minutes.



Starting point

x, y are distance with respect to the starting point

Let x be the position of the east bound and y the position of the north bound car at time t in hours. Let z be the distance between the cars.

Given: $\frac{dx}{dt} = 30$ mph $\quad \frac{dy}{dt} = 45$ mph

Question: What is $\frac{dz}{dt}$ when $t = \frac{1}{4}$ hr?

From the geometry $z^2 = x^2 + y^2$

Relate the rates: $\frac{d}{dt} z^2 = \frac{d}{dt}(x^2+y^2)$

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\Rightarrow \quad \frac{dz}{dt} = \frac{2x \frac{dx}{dt} + 2y \frac{dy}{dt}}{2z} = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{z}$$

We need $x, y,$ and z when $t = \frac{1}{4}$ hr.

At $t = \frac{1}{4}$ hr $\quad x = 30 \text{ mph} \cdot \frac{1}{4} h = \frac{30}{4}$ mi

and $\quad y = 45 \text{ mph} \cdot \frac{1}{4} h = \frac{45}{4}$ mi

At this time

$$z^2 = x^2 + y^2 = \left(\frac{30}{4}\,mi\right)^2 + \left(\frac{45}{4}\,mi\right)^2$$

$$= \frac{30^2 + 45^2}{4^2}\,mi^2$$

$$z = \frac{\sqrt{30^2+45^2}}{4}\,mi$$

So when $t = \frac{1}{4}$ hr

$$\frac{dz}{dt} = \frac{\frac{30}{4}\,mi\left(30\,\frac{mi}{h}\right) + \frac{45}{4}\,mi\left(45\,\frac{mi}{h}\right)}{\frac{\sqrt{30^2+45^2}}{4}\,mi}$$

$$= \frac{\frac{30^2}{4}\,\frac{mi^2}{h} + \frac{45^2}{4}\,\frac{mi^2}{h}}{\frac{\sqrt{30^2+45^2}}{4}\ mi}$$

$$\frac{dz}{dt} = \frac{(30^2+45^2)}{4}\,\frac{mi^2}{h} \cdot \frac{4}{\sqrt{30^2+45^2}}\,\frac{1}{mi}$$

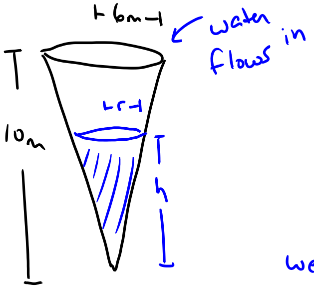
$$= \sqrt{30^2+45^2}\ \frac{mi}{h} \approx 54 \text{ mph}$$

General Approach to Solving Related Rates Prob.

- Identifty known and unknown quantities and assign variables.
- Create a diagram if possible.
- Use the diagram, physical science, and mathematics to connect known quantities to those being sought.
- **Relate the rates** of change using implicit differentiation.
- Substitute in known quantities and solve for desired quantities.

Example

A reservoir in the shape of an inverted right circular cone has height 10m and base radius 6m. If water is flowing into the reservoir at a constant rate of $50 m^3$/min. What is the rate at which the height of the water is increasing when the height is 5m?



Let r and h be the base radius and height of the water at time t in minutes. Take r, h in meters.

Let V be the volume of water @ time t.

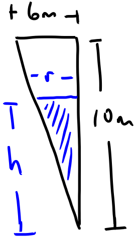
we're given $\frac{dV}{dt} = 50 \frac{m^3}{min}$

Question: Find $\frac{dh}{dt}$ when $h = 5m$.

From geometry $V = \frac{\pi}{3} r^2 h$

Let's connect r to h using similar triangles.

Take a cross section of the tank.



Using common ratios

$$\frac{r}{h} = \frac{6}{10} = \frac{3}{5} \Rightarrow r = \frac{3}{5} h$$

so

$$V = \frac{\pi}{3}\left(\frac{3}{5}h\right)^2 h = \frac{3\pi}{25} h^3$$

$$\frac{d}{dt}V = \frac{d}{dt}\left(\frac{3\pi}{25}h^3\right)$$

$$\frac{dV}{dt} = \frac{3\pi}{25}(3h^2)\frac{dh}{dt} = \frac{9\pi}{25}h^2\frac{dh}{dt}$$

when $h = 5m$ given $\frac{dV}{dt} = 50\,\frac{m^3}{min}$

$$50\,\frac{m^3}{min} = \frac{9\pi}{25}(5m)^2 \cdot \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = 50\,\frac{m^3}{min}\left(\frac{25}{9\pi}\right)\frac{1}{(5m)^2} =$$

$$= 50 \frac{m^3}{min} \cdot \frac{25}{9\pi} \frac{1}{25 m^2}$$

$$\frac{dh}{dt} = \frac{50}{9\pi} \frac{m}{min} \approx 1.8 \frac{m}{min}$$

The height is increasing @ about 1.8 $\frac{m}{min}$ when the height is 5m.

We had $\frac{r}{h} = \frac{3}{5} \Rightarrow r = \frac{3}{5}h$.

We can use this to find

$$\frac{d}{dt} r = \frac{d}{dt}\left(\frac{3}{5}h\right)$$

$$\frac{dr}{dt} = \frac{3}{5}\frac{dh}{dt}$$