Oct. 12 Math 1190 sec. 52 Fall 2016

Section 4.1: Related Rates

Motivating Example: A spherical balloon is being filled with air. Suppose that we know that the radius is increasing in time at a constant rate of 2 mm/sec. Can we determine the rate at which the surface area of the balloon is increasing at the moment that the radius is 10 cm?

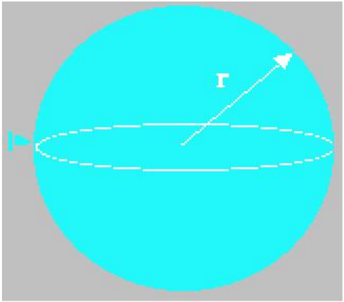


Figure: Spherical Balloon

Example Continued...

Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of time. Write an equation that relates

$$\frac{dS}{dt} \quad \text{to} \quad \frac{dr}{dt}.$$

We used the chain rule which says that

$$\frac{dS}{dt} = \frac{dS}{dr}\frac{dr}{dt}.$$

Since $\frac{dS}{dr} = 4\pi(2r)$, we get

$$\frac{dS}{dt} = 4\pi(2r)\frac{dr}{dt} = 8\pi r\frac{dr}{dt}.$$

Given this result, find the rate at which the surface area is changing when the radius is 10 cm.

We are told that r is increasing at a rate of 2 mm/sec. The derivative is the rate of change! So this tells us that

$$\frac{dr}{dt} = 2mm/sec.$$

Since r is increasing (given), the derivative above is positive 2. If r were decreasing, it would be negative 2. When $r = 10cm = 100mm$, we found

$$\frac{dS}{dt} = 8\pi(100mm)(2mm/sec) = 1600\pi\ mm^2/sec.$$

The surface area is increasing at a rate of 1600π mm^2 per second. (Note that the units make sense. mm^2 is a unit of area.)

Example

A right circular cone of height h and base radius r has volume

$$V = \frac{\pi}{3} r^2 h.$$

(a) Find $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$ if r is constant.

Using the chain rule (a.k.a implicit differentiation)

$$\frac{d}{dt} V = \frac{d}{dt}\left(\frac{\pi}{3} r^2 h\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3} r^2 \frac{dh}{dt}$$

since r, hence r^2, is constant

Example Continued...

$$V = \frac{\pi}{3} r^2 h$$

Question (b) Find $\frac{dV}{dt}$ in terms of $\frac{dr}{dt}$ if h is constant.

(a) $\frac{dV}{dt} = \frac{2\pi}{3} r \frac{dr}{dt}$

(b) $\frac{dV}{dt} = \frac{\pi}{3} r^2 \frac{dr}{dt}$

(c) $\frac{dV}{dt} = \frac{\pi}{3} r^2 h \frac{dr}{dt}$

(d) $\frac{dV}{dt} = \frac{2\pi}{3} rh \frac{dr}{dt}$

$$\frac{d}{dt} V = \frac{d}{dt}\left(\frac{\pi}{3} r^2 h\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3} h\,(2r) \cdot \frac{dr}{dt}$$

$$= \frac{2\pi}{3} rh \frac{dr}{dt}$$

And Continued Further...

$$V = \frac{\pi}{3} r^2 h$$

(c) Find $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$ and $\frac{dr}{dt}$ assuming neither r nor h is constant.

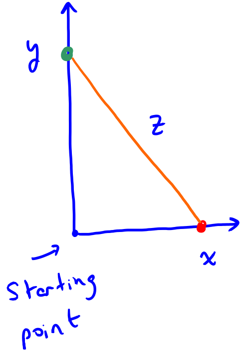
$$\frac{d}{dt} V = \frac{d}{dt}\left(\frac{\pi}{3} \underbrace{r^2 h}_{\text{product}}\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left(r^2 \frac{dh}{dt} + h\,(2r)\frac{dr}{dt}\right)$$

$$= \frac{\pi}{3} r^2 \frac{dh}{dt} + \frac{2\pi}{3} r h \frac{dr}{dt}$$

Example

Two cars leave from the same place. One heads due east at 30 mph, the other heads due north at a rate of 45 mph. At what rate is the distance between them changing after 15 minutes.



Let x be the distance of the east bound car and y the distance of the north bound car from the starting point in miles at the time t in hours.

Let z be the straight line distance between the cars (also in miles).

Given: $\frac{dx}{dt} = 30$ mph and $\frac{dy}{dt} = 45$ mph

Question: What is $\frac{dz}{dt}$ when $t = \frac{1}{4}$ hr?

From the geometry

$$z^2 = x^2 + y^2$$

Take the derivative

$$\frac{d}{dt} z^2 = \frac{d}{dt}(x^2+y^2)$$

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$\Rightarrow \quad \frac{dz}{dt} = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{z}$$

When $t = \frac{1}{4}$ hr $\quad x = 30 \, \frac{mi}{hr} \cdot \frac{1}{4} \, hr = \frac{30}{4} \, mi$

and $\quad y = 45 \, \frac{mi}{hr} \cdot \frac{1}{4} \, hr = \frac{45}{4} \, mi$

and $\quad z^2 = \left(\frac{30}{4} \, mi\right)^2 + \left(\frac{45}{4} \, mi\right)^2 = \frac{30^2 + 45^2}{4^2} \, mi^2$

$$\Rightarrow \quad z = \frac{\sqrt{30^2 + 45^2}}{4} \, mi$$

So when $t = \frac{1}{4}$ hr

$$\frac{dz}{dt} = \frac{\left(\frac{30}{4}\,mi\right) 30\,\frac{mi}{h} + \left(\frac{45}{4}\,mi\right) 45\,\frac{mi}{h}}{\frac{\sqrt{30^2+45^2}}{4}\,mi}$$

$$= \frac{\frac{30^2}{4}\,mi^2/h + \frac{45^2}{4}\,\frac{mi^2}{h}}{\frac{\sqrt{30^2+45^2}}{4}\,mi}$$

$$= \left(\frac{30^2}{4} + \frac{45^2}{4}\right)\frac{mi^2}{hr} \cdot \frac{4}{\sqrt{30^2+45^2}}\,\frac{1}{mi}$$

$$\frac{dz}{dt} = \sqrt{30^2+45^2}\ \frac{mi}{hr} \approx 54\ \frac{mi}{hr}$$

The distance between them is increasing at about 54 mph.

Note

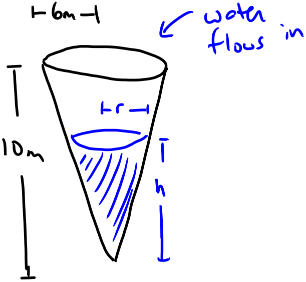
$$\frac{p}{\sqrt{p}} = \frac{(\sqrt{p})^2}{\sqrt{p}} = \frac{\sqrt{p}\,\cancel{\sqrt{p}}}{\cancel{\sqrt{p}}} = \sqrt{p}$$

General Approach to Solving Related Rates Prob.

- Identifty known and unknown quantities and assign variables.
- Create a diagram if possible.
- Use the diagram, physical science, and mathematics to connect known quantities to those being sought.
- **Relate the rates** of change using implicit differentiation.
- Substitute in known quantities and solve for desired quantities.

Example

A reservoir in the shape of an inverted right circular cone has height 10m and base radius 6m. If water is flowing into the reservoir at a constant rate of $50m^3$/min. What is the rate at which the height of the water is increasing when the height is 5m?



Let r and h be the base radius and height of the water in meters at the time t in minutes.

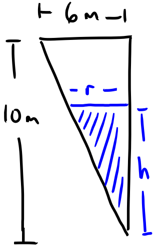
If the Volume of water is V then

we're given $\frac{dV}{dt} = 50 \frac{m^3}{min}$.

Question: What is $\frac{dh}{dt}$ when $h = 5m$?

From geometry $V = \frac{\pi}{3} r^2 h$.

We can write r in terms of h using similar triangles. Take a cross section.



Using fixed ratios

$$\frac{r}{h} = \frac{6}{10} = \frac{3}{5} \Rightarrow r = \frac{3}{5} h$$

so

$$V = \frac{\pi}{3}\left(\frac{3}{5} h\right)^2 h = \frac{3\pi}{25} h^3$$

Take the derivative

$$\frac{d}{dt} V = \frac{d}{dt}\left(\frac{3\pi}{25} h^3\right)$$

$$\frac{dV}{dt} = \frac{3\pi}{25}\, 3h^2 \cdot \frac{dh}{dt}$$

When $h = 5m$

$$50\ \frac{m^3}{min} = \frac{9\pi}{25}(5m)^2 \cdot \frac{dh}{dt}$$

$$= \frac{9\pi}{2\cancel{5}}(2\cancel{5}\, m^2)\frac{dh}{dt}$$

$$\frac{dh}{dt} = 50 \frac{m^3}{min} \frac{1}{9\pi m^2} = \frac{50}{9\pi} \frac{m}{min}$$

$$\approx 1.8 \frac{m}{min}$$

The height is increasing at about 1.8 m/min.