October 12 Math 2306 sec 51 Fall 2015

Section 4.9: Solving a System by Elimination

Recall: A linear system of ODE's is a collection of two or more linear ODE's with two or more dependent variables.

A first order, constant coefficient system IVP has the form

$$\begin{aligned}\frac{dx}{dt} &= a_{11}x + a_{12}y + f(t), \quad x(t_0) = x_0\\ \frac{dy}{dt} &= a_{21}x + a_{22}y + g(t), \quad y(t_0) = y_0\end{aligned}$$

If $f(t) = g(t) = 0$, the system is *homogeneous*. Otherwise it is nonhomogeneous.

A solution to the ODE part will be a pair $(x(t), y(t))$ containing 2-parameters (shared by the pair).

Operator Notation

Using the notation $D^n x = \frac{d^n x}{dt^n}$, the previous system may be expressed as

$$\begin{aligned} Dx &= a_{11}x + a_{12}y + f(t), \quad x(t_0) = x_0 \\ Dy &= a_{21}x + a_{22}y + g(t), \quad y(t_0) = y_0 \end{aligned}$$

or for even greater convenience

$$\begin{array}{rcrcll} (D - a_{11})x & - & a_{12}y & = & f(t), & x(t_0) = x_0 \\ -a_{21}x & + & (D - a_{22})y & = & g(t), & y(t_0) = y_0 \end{array}$$

Solving a System by Elimination

Remark: The current method is for linear systems with constant coefficients only.

- Write the system using the operator notation. Line up like variables so that the system appears as an algebraic system.
- Eliminate variables using standard operations. Keep in mind that "multiplication" by D is differentiation.
- Obtain an equation (or equations) in each variable separately, and solve using any applicable method.
- Use back substitution as needed to obtain solutions for all dependent variables.

Solve the System by Elimination

$$\begin{aligned}\frac{dx}{dt} &= 4x+7y\\ \frac{dy}{dt} &= x-2y\end{aligned} \Rightarrow \begin{aligned} Dx &= 4x+7y\\ Dy &= x-2y\end{aligned} \Rightarrow \begin{aligned} Dx-4x-7y &= 0\\ -x+Dy+2y &= 0\end{aligned}$$

$$(D-4)x - 7y = 0$$
$$-x + (D+2)y = 0$$

"Multiply" equation 2 by $D-4$

$$(D-4)x - 7y = 0$$
$$-(D-4)x + (D-4)(D+2)y = (D-4)0 = 0$$

add

$$-7y + (D-4)(D+2)y = 0$$

$$-7y + (D^2 - 4D + 2D - 8)y = 0 \Rightarrow (D^2 - 2D - 15)y = 0$$

$$-7y + (D^2 - 2D - 8)y = 0 \qquad y'' - 2y' - 15y = 0.$$

2nd order, constant coefficient, homogeneous eqn.

$$m^2 - 2m - 15 = 0 \Rightarrow (m-5)(m+3) = 0$$

$$m_1 = 5$$

$$m_2 = -3$$

$$y = c_1 e^{5t} + c_2 e^{-3t}$$

We need to solve for x. From the original 2^{nd} equation

$$x = \frac{dy}{dt} + 2y = 5c_1 e^{5t} - 3c_2 e^{-3t} + 2\left(c_1 e^{5t} + c_2 e^{-3t}\right)$$

$$= 7c_1 e^{5t} - c_2 e^{-3t}$$

The solution is

$$x = 7c_1 e^{5t} - c_2 e^{-3t}$$

$$y = c_1 e^{5t} + c_2 e^{-3t}$$

Solve the IVP by Elimination

$$\begin{aligned} x' - y &= 12t, & x(0) &= 4 \\ y' + x &= 2, & y(0) &= 3 \end{aligned}$$

Solve the DE's

$$Dx - y = 12t$$

$$x + Dy = 2$$

"Multiply" top eqn. by D

$$D^2x - Dy = D(12t) = 12$$

$$x + Dy = 2$$

add

$$D^2x + x = 14$$

$$x'' + x = 14$$

get x_c: $\quad x'' + x = 0 \quad m^2 + 1 = 0 \Rightarrow m = \pm i$

$\alpha = 0, \; \beta = 1$

$$x_c = c_1 \cos t + c_2 \sin t$$

get x_p: $\quad x_p = A$ (Method of Undetermined coefficients)

$x_p' = 0$

$x_p'' = 0$

$$x_p'' + x_p = 14 \Rightarrow 0 + A = 14 \Rightarrow A = 14$$

So $x(t) = c_1 \cos t + c_2 \sin t + 14$

From the original 1st equation

$$y = x' - 12t = -c_1 \sin t + c_2 \cos t - 12t$$

The general solution to the DEs is

$$x = c_1 \cos t + c_2 \sin t + 14$$

$$y = c_2 \cos t - c_1 \sin t - 12t$$

Apply $x(0) = 4$, $y(0) = 3$

$$x(0) = C_1 \cos 0 + C_2 \sin 0 + 14 = 4$$

$$C_1 + 14 = 4 \Rightarrow \quad C_1 = -10$$

$$y(0) = C_2 \cos 0 - C_1 \sin 0 - 12 \cdot 0 = 3$$

$$C_2 = 3$$

The solution to the IVP is

$$x = -10\cos t + 3\sin t + 14$$

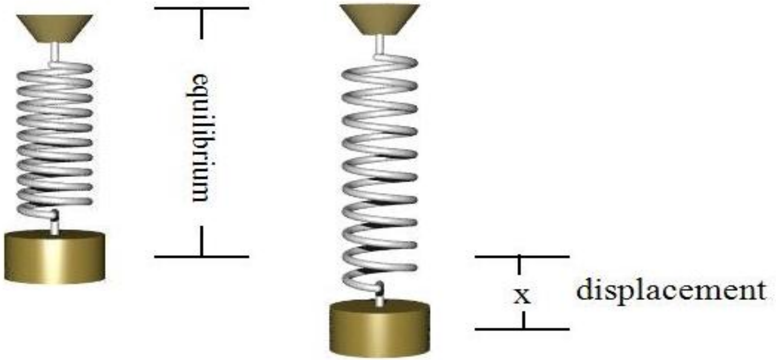
$$y = 3\cos t + 10\sin t - 12t$$

Section 5.1.1: Free Undamped Spring/Mass Systems

We consider a flexible spring from which a mass is suspended. In the absence of any damping forces (e.g. friction, a dash pot, etc.), and free of any external driving forces, any initial displacement or velocity imparted will result in **free, undamped motion**–a.k.a. **simple harmonic motion**.

Harmonic Motion gif

Building an Equation: Hooke's Law



At equilibrium, displacement $x(t) = 0$.

Hooke's Law: $F_{\text{spring}} = k\,x$

Figure: In the absence of any displacement, the system is at equilibrium. Displacement $x(t)$ is measured from equilibrium $x(t) = 0$.

Building an Equation: Hooke's Law

Newton's Second Law: $F = ma$ Force = mass times acceleration

$$a = \frac{d^2x}{dt^2} \quad \Longrightarrow \quad F = m\frac{d^2x}{dt^2}$$

Hooke's Law: $F = kx$ Force exerted by the spring is proportional to displacement
The force imparted by the spring opposes the direction of motion.

$$m\frac{d^2x}{dt^2} = -kx \quad \Longrightarrow \quad x'' + \omega^2 x = 0 \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}$$

Convention We'll Use: Down will be positive ($x > 0$), and up will be negative ($x < 0$).

Simple Harmonic Motion

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

called the **equation of motion.** Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$[1]
- the circular frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

[1] Various authors call f the natural frequency and others use this term for ω.

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by (for a sine representation)

$$\sin\phi = \frac{x_0}{A}, \quad \text{with} \quad \cos\phi = \frac{x_1}{\omega A}.$$

Derive $x_0 \cos(\omega t) + x_1/\omega sin(\omega t) = A \sin(\omega t + \phi)$

Wlt $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

$$x(t) = \sqrt{x_0^2 + \left(\frac{x_1}{\omega}\right)^2} \left(\frac{x_0}{\sqrt{x_0^2 + \left(\frac{x_1}{\omega}\right)^2}} \cos(\omega t) + \frac{x_1/\omega}{\sqrt{x_0^2 + (x_1/\omega)^2}} \sin(\omega t) \right)$$

Note

$$\Bigg(\underbrace{\frac{x_0}{\sqrt{x_0^2 + \left(\frac{x_1}{\omega}\right)^2}}}_{= \sin\phi}\Bigg)^2 + \Bigg(\underbrace{\frac{x_1/\omega}{\sqrt{x_0^2 + (x_1/\omega)^2}}}_{= \cos\phi}\Bigg)^2 = \frac{x_0^2 + (x_1/\omega)^2}{x_0^2 + (x_1/\omega)^2} = 1$$

$$x = A\left(\sin\phi \cos(\omega t) + \cos\phi \sin\omega t\right) = A \sin(\omega t + \phi)$$

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet below equilibrium with an initial upward velocity of 24 ft/sec. Find the equation of motion, the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32$ ft/sec^2.)

$$x(0) = x_0 = 4\text{ ft}, \quad x'(0) = x_1 = -24\ \frac{\text{ft}}{\text{sec}}$$

Find k: $\quad F = kx \quad \Rightarrow \quad 4\text{ lb} = k\,(0.5\text{ ft})$

$$\Rightarrow \quad k = 8\ \frac{\text{lb}}{\text{ft}}$$

Find m: $\quad W = mg \quad \Rightarrow \quad 4\text{ lb} = m\left(32\ \frac{\text{ft}}{\text{sec}^2}\right)$

$$\Rightarrow \quad m = \frac{1}{8}\ \frac{\text{lb sec}^2}{\text{ft}} = \frac{1}{8}\ \text{Slugs}$$

$$\omega^2 = \frac{k}{m} = \frac{8\ \frac{\text{lb}}{\text{ft}}}{\frac{1}{8}\ \frac{\text{lb sec}^2}{\text{ft}}} = 64\ {}^{1}/_{\text{sec}^2}$$

$$\Rightarrow \quad \omega = 8\ {}^{1}/\text{sec}.$$

$$x(t) = 4\cos(8t) - \frac{24}{8}\sin(8t)$$

$$\boxed{x(t) = 4\cos(8t) - 3\sin(8t)}$$

Amplitude

$$A = \sqrt{4^2 + (-3)^2} = 5$$

Period $\quad T = \frac{2\pi}{8} = \frac{\pi}{4}$

frequency $\quad f = \frac{4}{\pi}$

Phase Shift $\quad \sin\phi = \frac{4}{5}\ , \quad \cos\phi = \frac{-3}{5}$

Quad II angle

$$\phi = \cos^{-1}\left(\frac{-3}{5}\right) \approx 2.21 \text{ (radians)}$$