Oct 12 Math 2306 sec. 53 Fall 2018

Section 10: Variation of Parameters

For the second order equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation.

Variation of parameters provides a particular solution in the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x)$

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where

$$u_1 = \int \frac{-g(x)y_2(x)}{W}\,dx \quad \text{and} \quad u_2 = \int \frac{g(x)y_1(x)}{W}\,dx$$

and $W = W(y_1, y_2)(x)$ is the Wronskian of the solutions of the associated homogeneous equation.

Example:

Solve the ODE

$$x^2y'' + xy' - 4y = \ln x,$$

given that $y_c = c_1x^2 + c_2x^{-2}$ is the complementary solution.

$$y_1 = x^2 \text{ and } y_2 = x^{-2}$$

$$W = W(y_1, y_2)(x) = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = x^2(-2x^{-3}) - 2x\,(x^{-2})$$

$$= -2x^{-1} - 2x^{-1} = -4x^{-1}$$

Standard form

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = \frac{\ln x}{x^2}, \quad g(x) = \frac{\ln x}{x^2}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$u_1 = \int \frac{-g\, y_2}{W}\,dx = \int \frac{-\left(\frac{\ln x}{x^2}\right) x^{-2}}{-4x^{-1}}\,dx = \frac{1}{4}\int \frac{\ln x}{x^2}\cdot\frac{1}{x^2}\cdot x\,dx$$

$$= \frac{1}{4}\int x^{-3}\ln x\,dx$$

By parts

$u = \ln x \quad du = \frac{1}{x}\,dx$

$v = \frac{x^{-2}}{-2} \quad dv = x^{-3}\,dx$

$$= \frac{1}{4}\left(-\frac{x^{-2}}{2}\ln x - \int \frac{-x^{-2}}{2}\cdot\frac{1}{x}\,dx\right)$$

$$= \frac{1}{4}\left(-\frac{x^{-2}}{2}\ln x + \frac{1}{2}\int x^{-3}\,dx\right)$$

$$= \frac{1}{4}\left(-\frac{x^{-2}}{2}\ln x - \frac{1}{4}x^{-2}\right)$$

$$u_1 = -\frac{1}{8}x^{-2}\ln x - \frac{1}{16}x^{-2}$$

$$u_2 = \int \frac{g\,y_1}{W}\,dx = \int \frac{\frac{\ln x}{x^2}\cdot x^2}{-4x^{-1}}\,dx = \frac{-1}{4}\int x\ln x\,dx$$

Int by parts

$u = \ln x \qquad du = \frac{1}{x}\,dx$

$v = \frac{1}{2}x^2 \qquad dv = x\,dx$

$$= \frac{-1}{4}\left(\frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2\cdot\frac{1}{x}\,dx\right)$$

$$= \frac{-1}{4}\left(\frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx\right)$$

$$= \frac{-1}{4}\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)$$

$$u_1 = \frac{-1}{8}x^{-2}\ln x - \frac{1}{16}x^{-2} \qquad u_2 = \frac{-1}{8}x^{2}\ln x + \frac{1}{16}x^{2}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(\frac{-1}{8}x^{-2}\ln x - \frac{1}{16}x^{-2}\right)x^2 + \left(\frac{-1}{8}x^{2}\ln x + \frac{1}{16}x^{2}\right)x^{-2}$$

$$= \frac{-1}{8}\ln x - \frac{1}{16} + \frac{-1}{8}\ln x + \frac{1}{16}$$

$$= \frac{-1}{4}\ln x$$

The general solution is

$$y = C_1 x^2 + C_2 x^{-2} - \frac{1}{4}\ln x$$

Solve the IVP

$$x^2y'' + xy' - 4y = \ln x, \quad y(1) = -1, \quad y'(1) = 0$$

The general solution to the ODE is

$$y = c_1 x^2 + c_2 x^{-2} - \frac{1}{4}\ln x$$

$$y' = 2c_1 x - 2c_2 x^{-3} - \frac{1}{4}\cdot\frac{1}{x}$$

$$y(1) = c_1 1^2 + c_2 (1)^{-1} - \frac{1}{4}\ln(1) = -1$$

$$y'(1) = 2c_1\cdot 1 - 2c_2 1^{-3} - \frac{1}{4\cdot 1} = 0$$

$$C_1 + C_2 = -1$$

$$2C_1 - 2C_2 = \frac{1}{4}$$

$$2C_1 + 2C_2 = -2$$

$$4C_1 = \frac{-7}{4} \Rightarrow C_1 = \frac{-7}{16}$$

$$C_2 = -1 - C_1 = -1 + \frac{7}{16} = \frac{-9}{16}$$

The solution to the IVP is

$$\boxed{y = \frac{-7}{16}x^2 - \frac{9}{16}x^{-2} - \frac{1}{4}\ln x}$$