October 12 Math 2306 sec 54 Fall 2015

Section 4.9: Solving a System by Elimination

Recall: A linear system of ODE's is a collection of two or more linear ODE's with two or more dependent variables.

A first order, constant coefficient system IVP has the form

$$\begin{aligned}\frac{dx}{dt} &= a_{11}x + a_{12}y + f(t), \quad x(t_0) = x_0\\ \frac{dy}{dt} &= a_{21}x + a_{22}y + g(t), \quad y(t_0) = y_0\end{aligned}$$

If $f(t) = g(t) = 0$, the system is *homogeneous*. Otherwise it is nonhomogeneous.

A solution to the ODE part will be a pair $(x(t), y(t))$ containing 2-parameters (shared by the pair).

Operator Notation

Using the notation $D^n x = \frac{d^n x}{dt^n}$, the previous system may be expressed as

$$\begin{aligned} Dx &= a_{11}x + a_{12}y + f(t), \quad x(t_0) = x_0 \\ Dy &= a_{21}x + a_{22}y + g(t), \quad y(t_0) = y_0 \end{aligned}$$

or for even greater convenience

$$\begin{array}{rcrcll} (D - a_{11})x & - & a_{12}y & = & f(t), & x(t_0) = x_0 \\ -a_{21}x & + & (D - a_{22})y & = & g(t), & y(t_0) = y_0 \end{array}$$

Solving a System by Elimination

Remark: The current method is for linear systems with constant coefficients only.

- Write the system using the operator notation. Line up like variables so that the system appears as an algebraic system.
- Eliminate variables using standard operations. Keep in mind that "multiplication" by D is differentiation.
- Obtain an equation (or equations) in each variable separately, and solve using any applicable method.
- Use back substitution as needed to obtain solutions for all dependent variables.

Solve the System by Elimination

$$\frac{dx}{dt} = 4x+7y$$
$$\frac{dy}{dt} = x-2y$$

We rewrote this as

$$\begin{array}{rcrcl} (D-4)x & - & 7y & = & 0 \\ -x & + & (D+2)y & = & 0, \end{array}$$

eliminated x, and obtained the equation

$$y''-2y'-15y=0, \quad \text{with solution} \quad y=c_1e^{5t}+c_2e^{-3t}.$$

We still need to find $x(t)$.

From the original 2^{nd} equation

$$x = \frac{dy}{dt} + 2y$$

$$= 5c_1 e^{5t} - 3c_2 e^{-3t} + 2\left(c_1 e^{5t} + c_2 e^{-3t}\right)$$

$$= 7c_1 e^{5t} - c_2 e^{-3t}$$

The general solution to the system is

$$x = 7c_1 e^{5t} - c_2 e^{-3t}$$

$$y = c_1 e^{5t} + c_2 e^{-3t}.$$

Solve the IVP by Elimination

$$\begin{aligned} x' - y &= 12t, \quad x(0) = 4 \\ y' + x &= 2, \quad\ \ y(0) = 3 \end{aligned}$$

Solve the DEs first

$$Dx - y = 12t$$
$$x + Dy = 2$$

"Multiply" 1st eqn by D

$$D^2x - Dy = D(12t) = 12$$
$$x + Dy = 2$$

add

$$D^2x + x = 14$$

We have $x'' + x = 14$ 2nd order constant coeff. nonhomogeneous

Find x_c: $x'' + x = 0$

$$m^2 + 1 = 0 \implies m = \pm i \quad \alpha = 0, \ \beta = 1$$

$$x_c = C_1 \cos t + C_2 \sin t$$

Find x_p: Use Method of Undetermined coefficients

$$x_p = A$$

$x_p' = 0, \quad x_p'' = 0 \qquad x_p'' + x_p = 14$

$0 + A = 14$

So $\quad x(t) = c_1 \cos t + c_2 \sin t + 14$

From the original 1^{st} equation

$$y = x' - 12t$$

$$= -c_1 \sin t + c_2 \cos t - 12t$$

$$x = c_1 \cos t + c_2 \sin t + 14$$

$$y = -c_1 \sin t + c_2 \cos t - 12t$$

Apply $x(0) = 4$ $\quad$ $y(0) = 3$

$$x(0) = c_1 \cos 0 + c_2 \sin 0 + 14 = 4$$

$$c_1 + 14 = 4 \Rightarrow c_1 = -10$$

$$y(0) = -c_1 \sin 0 + c_2 \cos 0 - 12\cdot 0 = 3 \Rightarrow c_2 = 3$$

The solution to the IVP is

$$x = -10\cos t + 3\sin t + 14$$

$$y = 10\sin t + 3\cos t - 12t.$$

Section 5.1.1: Free Undamped Spring/Mass Systems

We consider a flexible spring from which a mass is suspended. In the absence of any damping forces (e.g. friction, a dash pot, etc.), and free of any external driving forces, any initial displacement or velocity imparted will result in **free, undamped motion**–a.k.a. **simple harmonic motion.**

Harmonic Motion gif

Building an Equation: Hooke's Law

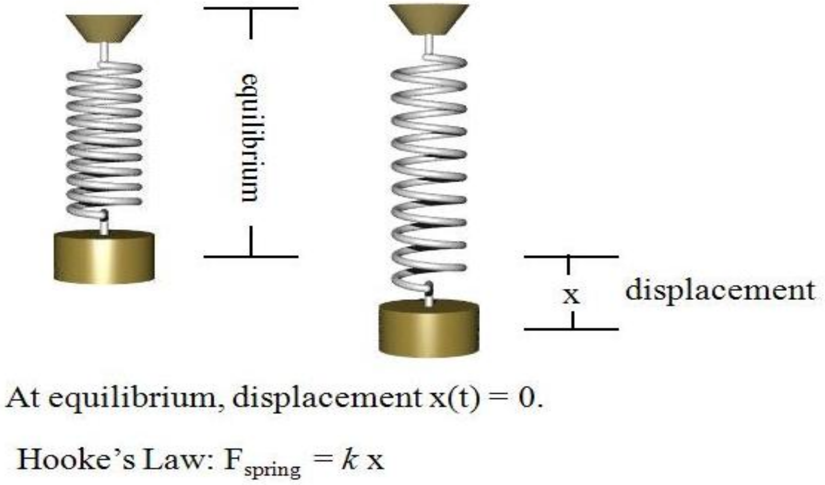

Figure: In the absence of any displacement, the system is at equilibrium. Displacement $x(t)$ is measured from equilibrium $x(t) = 0$.

Building an Equation: Hooke's Law

Newton's Second Law: $F = ma$ Force = mass times acceleration

$$a = \frac{d^2x}{dt^2} \quad \Longrightarrow \quad F = m\frac{d^2x}{dt^2}$$

Hooke's Law: $F = kx$ Force exerted by the spring is proportional to displacement
The force imparted by the spring opposes the direction of motion.

$$m\frac{d^2x}{dt^2} = -kx \quad \Longrightarrow \quad x'' + \omega^2 x = 0 \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}$$

Convention We'll Use: Down will be positive ($x > 0$), and up will be negative ($x < 0$).

Simple Harmonic Motion

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

called the **equation of motion.** Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$ [1]
- the circular frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

[1] Various authors call f the natural frequency and others use this term for ω.

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by (for a sine representation)

$$\sin\phi = \frac{x_0}{A}, \quad \text{with} \quad \cos\phi = \frac{x_1}{\omega A}.$$

Derive $x_0 \cos(\omega t) + x_1/\omega sin(\omega t) = A \sin(\omega t + \phi)$

$$A = \sqrt{x_0^2 + \left(\frac{x_1}{\omega}\right)^2}$$

Note: $\left(\frac{x_0}{\sqrt{x_0^2 + \left(\frac{x_1}{\omega}\right)^2}}\right)^2 + \left(\frac{x_1/\omega}{\sqrt{x_0^2 + \left(\frac{x_1}{\omega}\right)^2}}\right)^2 = \frac{x_0^2 + \left(\frac{x_1}{\omega}\right)^2}{x_0^2 + \left(\frac{x_1}{\omega}\right)^2} = 1$

Let $\sin\phi = \frac{x_0}{A}$, $\cos\phi = \frac{x_1/\omega}{A}$

$$x = A\left(\frac{x_0}{A}\cos(\omega t) + \frac{x_1/\omega}{A}\sin(\omega t)\right)$$

$$= A\left(\sin\phi \cos(\omega t) + \cos\phi \sin(\omega t)\right)$$

$$= A \sin(\omega t + \phi)$$

sum of angles formula for sine.

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet below equilibrium with an initial upward velocity of 24 ft/sec. Find the equation of motion, the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32$ ft/sec^2.)

$x(0) = x_0 = 4\ ft \qquad x'(0) = x_1 = -24\ \frac{ft}{sec}$

Find k: $\quad 4\,lb = k(0.5\,ft) \implies k = 8\ \frac{lb}{ft}$

Find m: $\quad$ Weight $W = mg \qquad 4\,lb = m\left(32\ \frac{ft}{sec^2}\right)$

$$m = \frac{1}{8}\ \frac{lb\,sec^2}{ft} = \frac{1}{8}\ \text{slugs}$$

$$\omega^2 = \frac{k}{m} = \frac{8\ \frac{lb}{ft}}{\frac{1}{8}\ \frac{lb\ sec^2}{ft}} = 64\ \frac{1}{sec^2}$$

$$\omega = 8\ {}^{1}/sec$$

$$x = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

$$x = 4\cos(8t) - \frac{24}{8} \sin(8t)$$

$$\boxed{x = 4\cos(8t) - 3\sin(8t)}$$

Period : $T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$

frequency : $f = \frac{4}{\pi}$

Amplitude : $A = \sqrt{4^2 + (-3)^2} = 5$

Phase Shift : $\sin\phi = \frac{x_0}{A} = \frac{4}{5}$, $\cos\phi = \frac{\frac{x_1}{\omega}}{A} = \frac{-3}{5}$

ϕ is a quad II angle since $\sin\phi > 0$ and

$\cos\phi < 0$

$$\phi = \cos^{-1}\left(\frac{-3}{5}\right) \approx 2.21 \approx 126.9^\circ$$

radian

$$x(t) = 5\sin(8t + 2.21)$$