Section 4.6: Rank

Remarks

- We can naturally associate three vector spaces with an \( m \times n \) matrix \( A \). Row \( A \) and Nul \( A \) are subspaces of \( \mathbb{R}^n \) and Col \( A \) is a subspace of \( \mathbb{R}^m \).

- Careful! The rows of the rref do span Row \( A \). But we go back to the columns in the original matrix to get vectors that span Col \( A \). (Get a basis for Col \( A \) from \( A \) itself!)

- Careful Again! Just because the first three rows of the rref span Row \( A \) does not mean the first three rows of \( A \) span Row \( A \). (Get a basis for Row \( A \) from the rref!)
Remarks

- Row operations preserve row space, but change linear dependence relations of rows. Row operations change column space, but preserve linear dependence relations of columns.

- Another way to obtain a basis for Row $A$ is to take the transpose $A^T$ and do row operations. We have the following relationships:

  \[ \text{Col } A = \text{Row } A^T \quad \text{and} \quad \text{Row } A = \text{Col } A^T. \]

- The dimension of the null space is called the **nullity**.
Rank

**Definition:** The rank of a matrix $A$ (denoted rank $A$) is the dimension of the column space of $A$.

**Theorem:** For $m \times n$ matrix $A$, $\dim \text{Col } A = \dim \text{Row } A = \text{rank } A$. Moreover

$$\text{rank } A + \dim \text{Nul } A = n.$$ 

**Note:** This theorem states the rather obvious fact that

$$\left\{ \text{number of pivot columns} \right\} + \left\{ \text{number of non-pivot columns} \right\} = \left\{ \text{total number of columns} \right\}.$$
Examples

(1) $A$ is a $5 \times 4$ matrix with rank $A = 4$. What is dim Nul $A$?

Here $n = 4$

\[ \text{rank } A + \text{dim Nul } A = n \]
\[ 4 + \text{dim Nul } A = 4 \]
\[ \text{dim Nul } A = 0 \]

What does this mean about the equation $A\vec{x} = \vec{0}$?

It has only the trivial solution.
Examples

(2) Suppose $A$ is $7 \times 5$ and $\dim \text{Col} \ A = 2$. Determine the nullity\(^1\) of $A$, the rank $A^T$, and the nullity of $A^T$.

For $A$, $n = 5$.

\[
\text{rank } A + \dim \text{Nul } A = 5
\]

\[
\text{rank } = \dim \text{Col } A
\]

\[
2 + \dim \text{Nul } A = 5 \implies \text{nullity of } A
\]

is 3.

\[
\text{Col } (A^T) = \text{Row } (A)
\]

\[
\dim \text{Col } (A^T) = \dim \text{Row } (A) = \dim \text{Col } (A) = 2
\]

$A^T$ is $5 \times 7$ so its "n" is 7.

\[
\text{rank } A^T + \text{nullity } A^T = 7
\]

\[
2 + \text{nullity } A^T = 7 \implies \text{nullity of } A^T
\]

is 5.

\(^1\)Nullity is another name for $\dim \text{Nul } A$. 

\[\text{nullity } A = \dim \text{Nul } A \]
Addendum to Invertible Matrix Theorem

Let $A$ be an $n \times n$ matrix. The following are equivalent to the statement that $A$ is invertible.

(m) The columns of $A$ form a basis for $\mathbb{R}^n$
(n) $\text{Col } A = \mathbb{R}^n$
(o) $\dim \text{Col } A = n$
(p) $\text{rank } A = n$
(q) $\text{Nul } A = \{0\}$
(r) $\dim \text{Nul } A = 0$
Section 6.1: Inner Product, Length, and Orthogonality

Recall: A vector $u$ in $\mathbb{R}^n$ can be considered an $n \times 1$ matrix. It follows that $u^T$ is a $1 \times n$ matrix

$$u^T = [u_1 \ u_2 \ \cdots \ u_n].$$

Definition: For vectors $u$ and $v$ in $\mathbb{R}^n$ we define the inner product of $u$ and $v$ (also called the dot product) by the matrix product

$$u^T v = [u_1 \ u_2 \ \cdots \ u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Note that this product produces a scalar. It is sometimes called a scalar product.
Theorem (Properties of the Inner Product)

We’ll use the notation \( u \cdot v = u^T v \).

**Theorem:** For \( u, v \) and \( w \) in \( \mathbb{R}^n \) and real scalar \( c \)

(a) \( u \cdot v = v \cdot u \)

(b) \( (u + v) \cdot w = u \cdot w + v \cdot w \)

(c) \( c(u \cdot v) = (cu) \cdot v = u \cdot (cv) \)

(d) \( u \cdot u \geq 0 \), with \( u \cdot u = 0 \) if and only if \( u = 0 \).
The Norm

The property \( \mathbf{u} \cdot \mathbf{u} \geq 0 \) means that \( \sqrt{\mathbf{u} \cdot \mathbf{u}} \) always exists as a real number.

**Definition:** The norm of the vector \( \mathbf{v} \) in \( \mathbb{R}^n \) is the nonnegative number denoted and defined by

\[
\| \mathbf{v} \| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}
\]

where \( v_1, v_2, \ldots, v_n \) are the components of \( \mathbf{v} \).

As a directed line segment, the norm is the same as the **length**.
Norm and Length

**Figure:** In $\mathbb{R}^2$ or $\mathbb{R}^3$, the norm corresponds to the classic geometric property of length.

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**Figure:** In $\mathbb{R}^2$ or $\mathbb{R}^3$, the norm corresponds to the classic geometric property of length.
Unit Vectors and Normalizing

Theorem: For vector $\mathbf{v}$ in $\mathbb{R}^n$ and scalar $c$

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}\|.$$  

Definition: A vector $\mathbf{u}$ in $\mathbb{R}^n$ for which $\|\mathbf{u}\| = 1$ is called a unit vector.

Remark: Given any nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$, we can obtain a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$  

This process, of dividing out the norm, is called normalizing the vector $\mathbf{v}$. 
Example

Show that $\mathbf{v}/\|\mathbf{v}\|$ is a unit vector.

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{v}\|} \mathbf{v}$$

so

$$\|\frac{1}{\|\mathbf{v}\|} \mathbf{v}\| = \left| \frac{1}{\|\mathbf{v}\|} \right| \|\mathbf{v}\|$$

$$= \frac{1}{\|\mathbf{v}\|} \|\mathbf{v}\|$$

$$= \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|} = 1$$

but $\|\mathbf{v}\| > 0$
Example

Find a unit vector in the direction of \( \mathbf{v} = (1, 3, 2) \).

\[
\| \mathbf{v} \|^2 = 1^2 + 3^2 + 2^2 = 1 + 9 + 4 = 14
\]

A unit vector in the direction of \( \mathbf{v} \) is

\[
\frac{1}{\| \mathbf{v} \|} \mathbf{v} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{bmatrix}
\]
Distance in $\mathbb{R}^n$

**Definition:** For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the **distance** between $\mathbf{u}$ and $\mathbf{v}$ is denoted and defined by

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \| \mathbf{u} - \mathbf{v} \|.$$ 

**Example:** Find the distance between $\mathbf{u} = (4, 0, -1, 1)$ and $\mathbf{v} = (0, 0, 2, 7)$.

$$\mathbf{u} - \mathbf{v} = (4, 0, -1, 1) - (0, 0, 2, 7) = (4, 0, -3, -6)$$

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \sqrt{4^2 + 0^2 + (-3)^2 + (-6)^2} = \sqrt{16 + 9 + 36} = \sqrt{61}.$$
Orthogonality

Definition: Two vectors are \( \mathbf{u} \) and \( \mathbf{v} \) orthogonal if \( \mathbf{u} \cdot \mathbf{v} = 0. \)

Figure: Note that two vectors are perpendicular if \( || \mathbf{u} - \mathbf{v} || = || \mathbf{u} + \mathbf{v} || \)
Orthogonal and Perpendicular

Show that \( \| \mathbf{u} - \mathbf{v} \| = \| \mathbf{u} + \mathbf{v} \| \) if and only if \( \mathbf{u} \cdot \mathbf{v} = 0 \).

Note

\[
\| \mathbf{u} - \mathbf{v} \|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = (\mathbf{u} - \mathbf{v}) \cdot \mathbf{u} - (\mathbf{u} - \mathbf{v}) \cdot \mathbf{v}
\]

\[
= \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}
\]

\[
= \| \mathbf{u} \|^2 - 2 \mathbf{u} \cdot \mathbf{v} + \| \mathbf{v} \|^2
\]

Similarly

\[
\| \mathbf{u} + \mathbf{v} \|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})
\]

\[
= \| \mathbf{u} \|^2 + 2 \mathbf{u} \cdot \mathbf{v} + \| \mathbf{v} \|^2
\]

Note: If \( \| \mathbf{u} - \mathbf{v} \|^2 = \| \mathbf{u} + \mathbf{v} \|^2 \) then \( \| \mathbf{u} - \mathbf{v} \| = \| \mathbf{u} + \mathbf{v} \| \).
If \( \| \mathbf{u} - \mathbf{v} \| = \| \mathbf{u} + \mathbf{v} \| \), then \( \| \mathbf{u} - \mathbf{v} \|^2 = \| \mathbf{u} + \mathbf{v} \|^2 \)

so

\[
\| \mathbf{u} \|^2 - 2 \mathbf{u} \cdot \mathbf{v} + \| \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + 2 \mathbf{u} \cdot \mathbf{v} + \| \mathbf{v} \|^2
\]

\[\Rightarrow 0 = 4 \mathbf{u} \cdot \mathbf{v} \Rightarrow \mathbf{u} \cdot \mathbf{v} = 0\]

If \( \mathbf{u} \cdot \mathbf{v} = 0 \), then

\[\| \mathbf{u} - \mathbf{v} \| = \| \mathbf{u} \|^2 - 2 \cdot 0 + \| \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2\]

and

\[\| \mathbf{u} + \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + 2 \cdot 0 + \| \mathbf{v} \|^2 = \| \mathbf{u} \|^2 + \| \mathbf{v} \|^2\]

Hence \( \| \mathbf{u} + \mathbf{v} \| = \| \mathbf{u} - \mathbf{v} \| \).