October 12 Math 3260 sec. 58 Fall 2017

Section 4.6: Rank

Remarks

- We can naturally associate three vector spaces with an $m \times n$ matrix A. Row A and Nul A are subspaces of $\mathbb{R}^n$ and Col A is a subspace of $\mathbb{R}^m$.

- Careful! The rows of the rref do span Row A. But we go back to the columns in the original matrix to get vectors that span Col A. (Get a basis for Col A from A itself!)

- Careful Again! Just because the first three rows of the rref span Row A **does not mean** the first three rows of A span Row A. (Get a basis for Row A from the rref!)

Remarks

- Row operations preserve row space, but change linear dependence relations of rows. Row operations change column space, but preserve linear dependence relations of columns.

- Another way to obtain a basis for Row A is to take the transpose A^T and do row operations. We have the following relationships:

$$\text{Col } A = \text{Row } A^T \quad \text{and} \quad \text{Row } A = \text{Col } A^T.$$

- The dimension of the null space is called the **nullity**.

Rank

Definition: The **rank** of a matrix A (denoted rank A) is the dimension of the column space of A.

Theorem: For $m \times n$ matrix A, dim Col A = dim Row A = rank A. Moreover

$$\text{rank}\, A + \dim \text{Nul}\, A = n.$$

Note: This theorem states the rather obvious fact that

$$\left\{ \begin{array}{c} \text{number of} \\ \text{pivot columns} \end{array} \right\} + \left\{ \begin{array}{c} \text{number of} \\ \text{non-pivot columns} \end{array} \right\} = \left\{ \begin{array}{c} \text{total number} \\ \text{of columns} \end{array} \right\}.$$

Examples

(1) A is a 5×4 matrix with rank $A = 4$. What is dim Nul A?

Here $n = 4$.

$$\operatorname{rank} A + \dim \operatorname{Nul} A = 4$$

$$4 + \dim \operatorname{Nul} A = 4 \Rightarrow \dim \operatorname{Nul} A = 0.$$

$$\operatorname{Nul}(A) = \{\vec{0}\}$$

What does this say about the equation

$$A\vec{x} = \vec{0}\ ?$$

It has only the trivial solution.

Examples

(2) Suppose A is 7×5 and $\dim \operatorname{Col} A = 2$. Determine the nullity[1] of A, the rank A^T, and the nullity of A^T.

For A, $n = 5$. $\quad \operatorname{rank} A = \dim \operatorname{Col} A = 2$

$\operatorname{rank} A + \dim \operatorname{Nul} A = 5 \Rightarrow 2 + \dim \operatorname{Nul} A = 5$

The nullity is $5 - 2 = 3$.

$\operatorname{rank} A^T = \dim \operatorname{Col} A^T = \dim \operatorname{Row} A = \dim \operatorname{Col} A = 2$.

For A^T, "n" is 7.

$\operatorname{rank} A^T + \dim \operatorname{Nul} A^T = 7 \Rightarrow \dim \operatorname{Nul} A^T = 5$

the nullity of A^T is 5.

[1] Nullity is another name for $\dim \operatorname{Nul} A$.

Addendum to Invertible Matrix Theorem

Let A be an $n \times n$ matrix. The following are equivalent to the statement that A is invertible.

(m) The columns of A form a basis for $\mathbb{R}^n$

(n) $\text{Col } A = \mathbb{R}^n$

(o) $\dim \text{Col } A = n$

(p) $\text{rank } A = n$

(q) $\text{Nul } A = \{\mathbf{0}\}$

(r) $\dim \text{Nul } A = 0$

Section 6.1: Inner Product, Length, and Orthogonality

Recall: A vector **u** in $\mathbb{R}^n$ can be considered an $n \times 1$ matrix. It follows that $\mathbf{u}^T$ is a $1 \times n$ matrix

$$\mathbf{u}^T = [u_1\ u_2\ \cdots\ u_n].$$

Definition: For vectors **u** and **v** in $\mathbb{R}^n$ we define the **inner product** of **u** and **v** (also called the **dot product**) by the **matrix product**

$$\mathbf{u}^T\mathbf{v} = [u_1\ u_2\ \cdots\ u_n]\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Note that this product produces a scalar. It is sometimes called a *scalar product*.

Theorem (Properties of the Inner Product)

We'll use the notation $\mathbf{u}\cdot\mathbf{v} \;=\; \mathbf{u}^T\mathbf{v}.$

Theorem: For $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ and real scalar c

(a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

(b) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$

(c) $c(\mathbf{u}\cdot\mathbf{v}) = (c\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(c\mathbf{v})$

(d) $\mathbf{u}\cdot\mathbf{u}\geq 0$, with $\mathbf{u}\cdot\mathbf{u}=0$ if and only if $\mathbf{u}=\mathbf{0}$.

The Norm

The property $\mathbf{u} \cdot \mathbf{u} \geq 0$ means that $\sqrt{\mathbf{u} \cdot \mathbf{u}}$ always exists as a real number.

Definition: The **norm** of the vector $\mathbf{v}$ in $\mathbb{R}^n$ is the nonnegative number denoted and defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

where $v_1, v_2, \ldots, v_n$ are the components of $\mathbf{v}$.

As a directed line segment, the norm is the same as the **length**.

Norm and Length

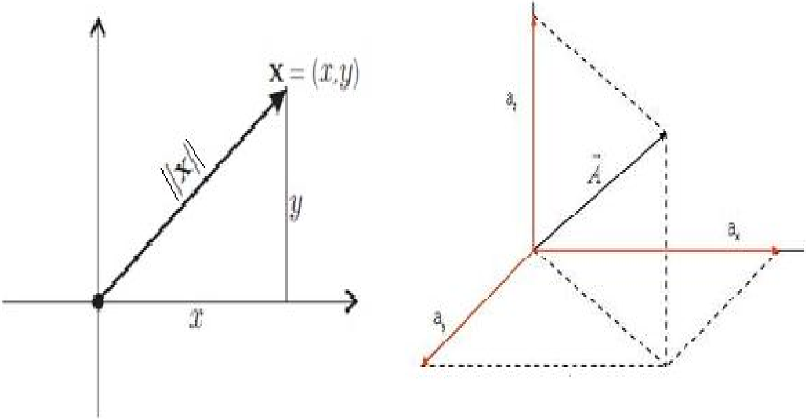

Figure: In $\mathbb{R}^2$ or $\mathbb{R}^3$, the norm corresponds to the classic geometric property of length.

Unit Vectors and Normalizing

Theorem: For vector $\mathbf{v}$ in $\mathbb{R}^n$ and scalar c

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}\|.$$

Definition: A vector $\mathbf{u}$ in $\mathbb{R}^n$ for which $\|\mathbf{u}\| = 1$ is called a **unit vector.**

Remark: Given any nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$, we can obtain a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

This process, of dividing out the norm, is called **normalizing** the vector $\mathbf{v}$.

Example

Show that $\mathbf{v}/\|\mathbf{v}\|$ is a unit vector.

$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\|\vec{v}\|}\vec{v} \qquad \text{and} \qquad \|\vec{v}\| > 0$$

$$\left\| \frac{1}{\|\vec{v}\|}\vec{v} \right\| = \left| \frac{1}{\|\vec{v}\|} \right| \|\vec{v}\|$$

$$= \frac{1}{\|\vec{v}\|}\|\vec{v}\| = \frac{\|\vec{v}\|}{\|\vec{v}\|} = 1$$

Example

Find a unit vector in the direction of $\mathbf{v} = (1, 3, 2)$.

$$\|\vec{v}\|^2 = 1^2 + 3^2 + 2^2 = 1 + 9 + 4 = 14$$

A unit vector in the direction of $\vec{v}$ is

$$\frac{1}{\sqrt{14}}\vec{v} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{bmatrix}$$

Distance in $\mathbb{R}^n$

Definition: For vectors **u** and **v** in $\mathbb{R}^n$, the **distance between u and v** is denoted and defined by

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Example: Find the distance between $\mathbf{u} = (4, 0, -1, 1)$ and $\mathbf{v} = (0, 0, 2, 7)$.

$\vec{u} - \vec{v} = (4,0,-1,1) - (0,0,2,7) = (4,0,-3,-6)$

$\text{dist}(\vec{u},\vec{v}) = \sqrt{4^2+0^2+(-3)^2+(-6)^2} = \sqrt{61}$

Orthogonality

Definition: Two vectors are $\mathbf{u}$ and $\mathbf{v}$ **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

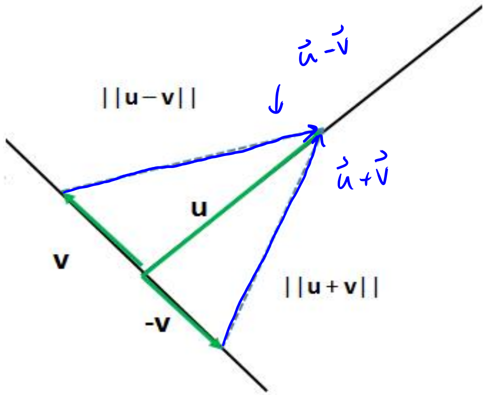

Figure: Note that two vectors are perpendicular if $\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{u} + \mathbf{v}\|$

Orthogonal and Perpendicular

Show that $\|\mathbf{u}-\mathbf{v}\| = \|\mathbf{u}+\mathbf{v}\|$ if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

Note $\|\vec{u}-\vec{v}\| = \|\vec{u}+\vec{v}\| \iff \|\vec{u}-\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2$

Observe

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot(\vec{u}-\vec{v}) - \vec{v}\cdot(\vec{u}-\vec{v})$$

$$= \vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$$

$$= \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$

Similarly

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$

If $\|\vec{u}-\vec{v}\| = \|\vec{u}+\vec{v}\|$, Then

$$\cancel{\|\vec{u}\|^2} - 2\vec{u}\cdot\vec{v} + \cancel{\|\vec{v}\|^2} = \cancel{\|\vec{u}\|^2} + 2\vec{u}\cdot\vec{v} + \cancel{\|\vec{v}\|^2}$$

$$0 = 4\vec{u}\cdot\vec{v} \implies \vec{u}\cdot\vec{v} = 0$$

If $\vec{u}\cdot\vec{v} = 0$, then

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 - 2\cdot 0 + \|\vec{v}\|^2$$

$$= \|\vec{u}\|^2 + 2\cdot 0 + \|\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2$$

$$\implies \|\vec{u}-\vec{v}\| = \|\vec{u}+\vec{v}\|.$$

The Pythagorean Theorem

Theorem: Two vectors **u** and **v** are orthogonal if and only if

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

This follows directly from

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$