Oct. 14 Math 1190 sec. 51 Fall 2016

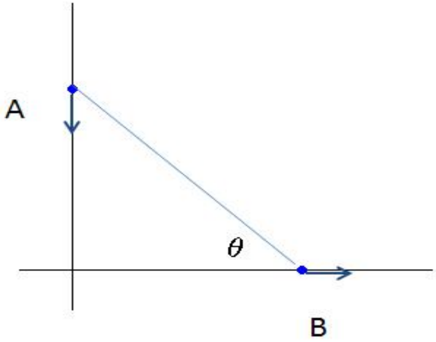
Section 4.1: Related Rates

General Approach to Solving Related Rates Problems:

- Identifty known and unknown quantities and assign variables.
- Create a diagram if possible.
- Use the diagram, physical science, and mathematics to connect known quantities to those being sought.
- **Relate the rates** of change using implicit differentiation.
- Substitute in known quantities and solve for desired quantities.

Let's Do One Together

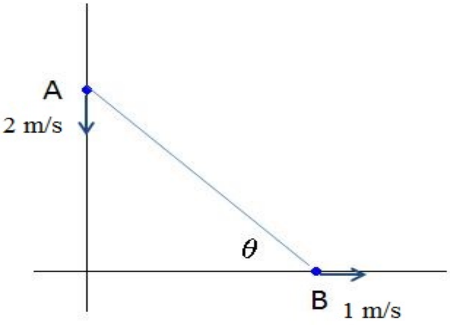
Pedestrians A and B are walking on straight streets that meet at right angles. A approaches the intersection at 2m/sec, and B moves away from the intersection at 1m/sec. Our goal is to determine the rate at which the angle θ shown in the diagram is changing when A is 10m from the intersection and B is 20 m from the intersection?



Question

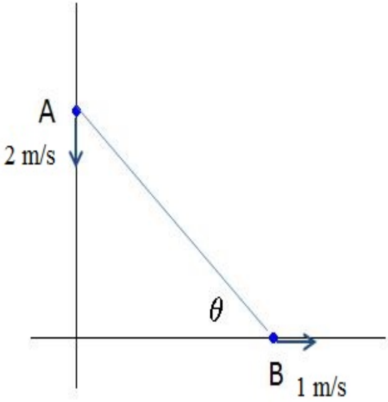
Let $A(t)$ be pedestrian A's position (distance to intersection), and $B(t)$ be pedestrian B's position. Let's make some observations:

(a) **True or False** A is decreasing. True
(b) **True or False** B is increasing. True



Question

From the diagram, which of the following are the rates of change of *A* and *B* (in m/s)?



(a) $\frac{dA}{dt} = -2$ m/s and $\frac{dB}{dt} = 1$ m/s

(b) $\frac{dA}{dt} = 2$ and $\frac{dB}{dt} = -1$

(c) $\frac{dA}{dt} = -2$ and $\frac{dB}{dt} = -1$

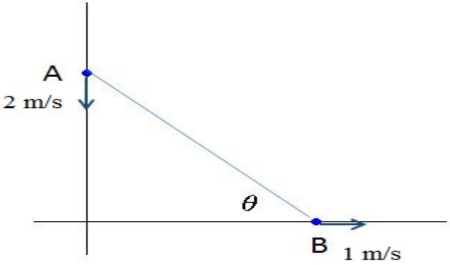
(d) $\frac{dA}{dt} = 2$ and $\frac{dB}{dt} = 1$

Relating the Rates

The pedestrians' positions and the intersection form a right triangle. So θ, *A*, and *B* are related by the equation

$$\tan\theta = \frac{A}{B}$$

Question: Use implicit differentiation to find an expression relating $\frac{d\theta}{dt}$ to the rates of *A* and *B*.



Question

The relation between the rates is given by

(a) $\dfrac{d\theta}{dt} = \dfrac{\frac{dA}{dt}B - A\frac{dB}{dt}}{B^2}$

(b) $\sec^2\left(\dfrac{d\theta}{dt}\right) = \dfrac{\frac{dA}{dt}}{\frac{dB}{dt}}$

(c) $\sec^2(\theta)\dfrac{d\theta}{dt} = \dfrac{\frac{dA}{dt}B - A\frac{dB}{dt}}{B^2}$

(d) $\sec^2(\theta)\dfrac{d\theta}{dt} = \dfrac{A}{B}\dfrac{dA}{dt} + \dfrac{A}{B}\dfrac{dB}{dt}$

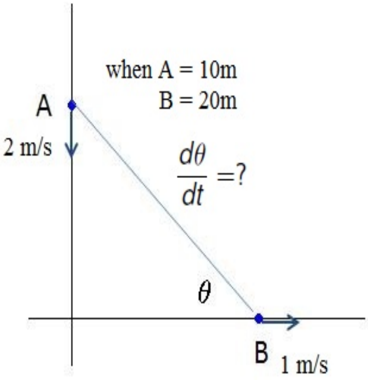
$\tan\theta = \dfrac{A}{B}$

$\dfrac{d}{dt}\tan\theta = \dfrac{d}{dt}\left(\dfrac{A}{B}\right)$

$\sec^2\theta \cdot \dfrac{d\theta}{dt} = \dfrac{\frac{dA}{dt}B - A\frac{dB}{dt}}{B^2}$

The Final Result

Determine the rate at which the angle θ shown in the diagram is changing when A is 10m from the intersection and B is 20 m from the intersection?

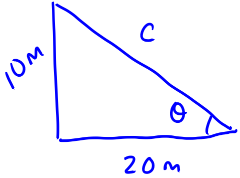


$$\sec^2\theta \, \frac{d\theta}{dt} = \frac{\frac{dA}{dt}B - A\frac{dB}{dt}}{B^2}$$

$$\frac{d\theta}{dt} = \frac{\frac{dA}{dt}B - A\frac{dB}{dt}}{B^2} \cdot \frac{1}{\sec^2\theta}$$

$$= \frac{\frac{dA}{dt}B - A\frac{dB}{dt}}{B^2} \cdot \cos^2\theta$$

When $A = 10\,m$, $B = 20\,m$, $\frac{dA}{dt} = -2\ \frac{m}{sec}$, $\frac{dB}{dt} = 1\ \frac{m}{sec}$.



$$c^2 = 10^2 + 20^2 = 500 \Rightarrow c = 10\sqrt{5}$$

Then $\cos\theta = \frac{20\,m}{10\sqrt{5}\,m} = \frac{2}{\sqrt{5}}$

$$\frac{d\theta}{dt} = \frac{\frac{dA}{dt}B - A\frac{dB}{dt}}{B^2}\cos^2\theta$$

at

$A = 10\,m$

$B = 20\,m$

$$\frac{d\theta}{dt} = \frac{-2\frac{m}{sec}\cdot 20m - 10m\cdot 1\frac{m}{sec}}{(20m)^2}\cdot\left(\frac{2}{\sqrt{5}}\right)^2$$

$$= \frac{(-40-10)\ \frac{m^2}{sec}}{400\ m^2} \cdot \frac{4}{5}$$

$$= \frac{-50}{400} \cdot \frac{4}{5} \cdot \frac{1}{sec}$$

$$= \frac{-10}{100}\ \frac{1}{sec} = \frac{-1}{10}\ \frac{1}{sec}$$

The angle is decreasing at a rate of $\frac{1}{10}$ radian per second at that moment.

Section 4.2: Maximum and Minimum Values; Critical Numbers

Definition: Let f be a function with domain D and let c be a number in D. Then $f(c)$ is

- **the absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D,
- **the absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D.

x-value y-value

Note that if an absolute minimum occurs at c, then $f(c)$ is the **absolute minimum value** of f. Similarly, if an absolute maximum occurs at c, then $f(c)$ is the **absolute maximum value** of f.

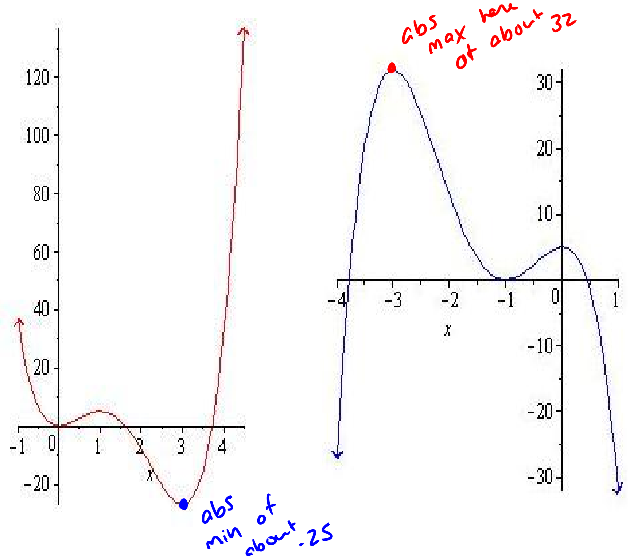


Figure: Graphically, an absolute minimum is the lowest point and an absolute maximum is the highest point.

Local Maximum and Minimum

Definition: Let f be a function with domain D and let c be a number in D. Then $f(c)$ is

- **a local minimum** value of f if $f(c) \leq f(x)$ for x *near** c
- **a local maximum** value of f if $f(c) \geq f(x)$ for x *near* c.

More precisely, to say that x is *near* c means that there exists an open interval containing c such that for all x in this interval the respective inequality holds.

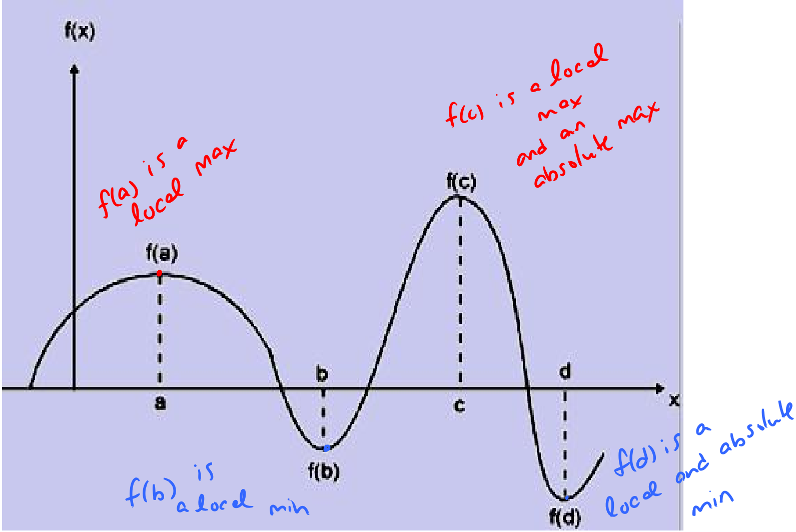


Figure: Graphically, local maxes and mins are *relative* high and low points.

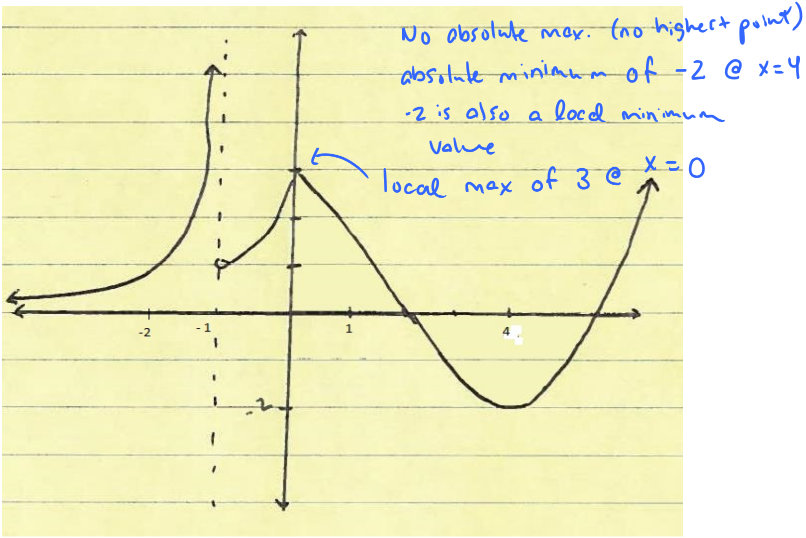


Figure: Identify local and absolute maxima and minima (if possible).

Terminology

Maxima—-plural of maximum

Minima—-plural of minimum

Extremum——is either a maximum or a minimum

Extrema——plural of extremum

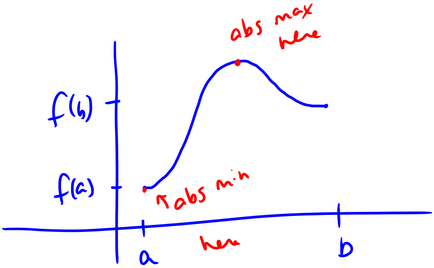
”**Global**” is another word for absolute.

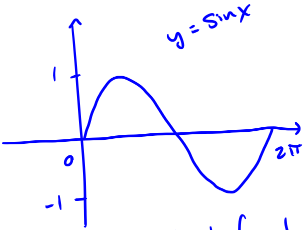
”**Relative**” is another word for local.

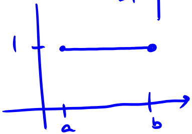
Extreme Value Theorem (EVT)

Suppose f is continuous on a closed interval $[a, b]$. Then f attains an absolute maximum value $f(c)$ and f attains an absolute minimum value $f(d)$ for some numbers c and d in $[a, b]$.

absolute max and min values can occur between a and b or at an end point a and/or b.







constant function every point gives abs max and abs min.

Fermat's Theorem

Note that the Extreme Value Theorem tells us that a continuous function is guaranteed to take an absolute maximum and absolute minimum on a closed interval. It does not provide a method for actually finding these values or where they occur. For that, the following theorem due to Fermat is helpful.

Theorem: If f has a local extremum at c and if $f'(c)$ exists, then

$$f'(c) = 0.$$

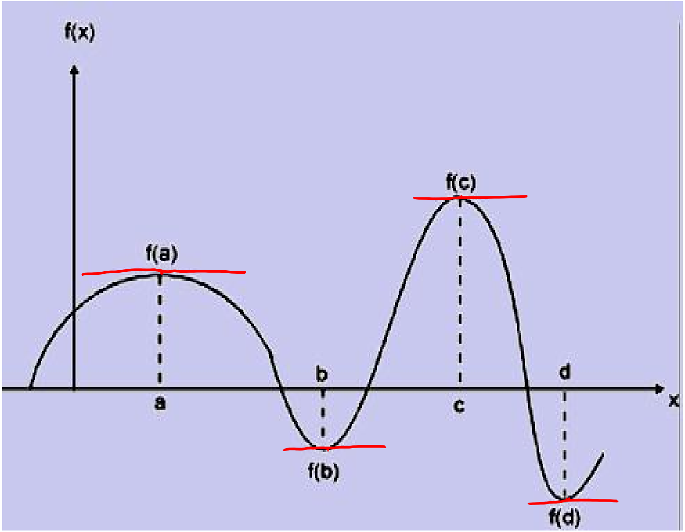
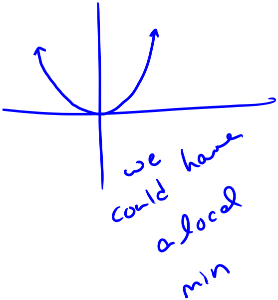


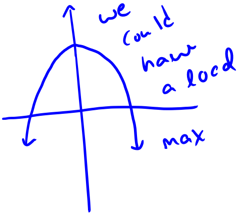
Figure: We note that at the local extrema, the tangent line would be horizontal.

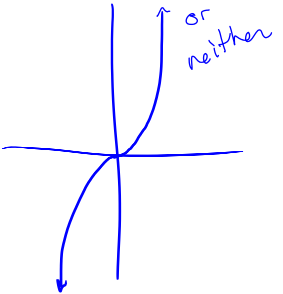
Is the Converse of our Theorem True?

Suppose a function f satisfies $f'(0) = 0$. Can we conclude that $f(0)$ is a local maximum or local minimum?

No.



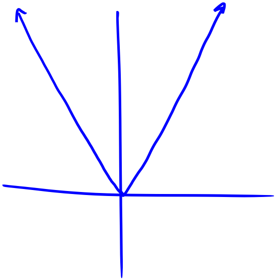




Does an extremum have to correspond to a horizontal tangent? No

Could $f(c)$ be a local extremum but have $f'(c)$ not exist? Yes

The classic example is $y = |x|$



It has a local (in fact global) minimum of zero @ $x=0$.

But y is not differentiable @ $x=0$.

Critical Number

Definition: A **critical number** of a function f is a number c in its domain such that either

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

Theorem: If f has a local extremum at c, then c is a critical number of f.

Some authors call critical numbers *critical points*.

Example

Find all of the critical numbers of the function.

(a) $f(x) = x^4 - 2x^2 + 5$

We need to know (1) where is $f'(x) = 0$ and

(2) where is $f'(x)$ undefined.

The domain is $(-\infty, \infty)$.

$f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1)$

$f'(x) = 0$ if $x=0$, $x=1$, or $x=-1$. $f'(x)$ is defined everywhere.

f has three critical numbers, 0, 1, and -1.