

Section 5.1.1: Free Undamped Spring/Mass Systems

Simple Harmonic Motion: Displacement $x(t)$ satisfies

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

called the **equation of motion**. Characteristics of the system include

- ▶ the period $T = \frac{2\pi}{\omega}$,
- ▶ the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$
- ▶ the circular frequency ω , and
- ▶ the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by (for a sine representation)

$$\sin \phi = \frac{x_0}{A}, \quad \text{with} \quad \cos \phi = \frac{x_1}{\omega A}.$$

Example

Recall down is positive
up is negative

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet below equilibrium with an initial upward velocity of 24 ft/sec. Find the equation of motion, the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32 \text{ ft/sec}^2$.)

We determined that $k = 8 \text{ lb/ft}$, $m = 1/8 \text{ slugs}$, and

$$x(t) = 4 \cos(8t) - 3 \sin(8t)$$

Period $T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$

frequency $f = \frac{4}{\pi}$

$$x(t) = 4 \cos(8t) - 3 \sin(8t)$$

Amplitude $A = \sqrt{4^2 + (-3)^2} = 5$

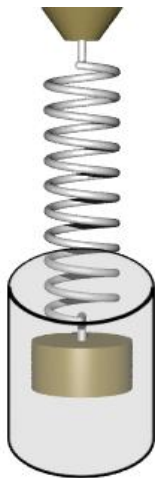
Phase shift: $\sin \phi = \frac{x_0}{A} = \frac{4}{5}$

$$\cos \phi = \frac{x_1/\omega}{A} = \frac{-3}{5}$$

$\sin \phi > 0$, $\cos \phi < 0$ ϕ is in quad II i.e. $\frac{\pi}{2} < \phi < \pi$

$$\phi = \cos^{-1}\left(\frac{-3}{5}\right) \approx 2.21 \quad (\text{rad.ans})$$

5.1.2: Free Damped Motion



fluid resists motion

$$F_{\text{damping}} = \beta \frac{dx}{dt}$$

$\beta > 0$ (by conservation of energy)

Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Force = Force of spring + Force of damping

$$m \frac{d^2 x}{dt^2} = -\beta \frac{dx}{dt} - kx \quad \implies \quad \frac{d^2 x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t} \left(c_1 e^{t\sqrt{\lambda^2 - \omega^2}} + c_2 e^{-t\sqrt{\lambda^2 - \omega^2}} \right)$$

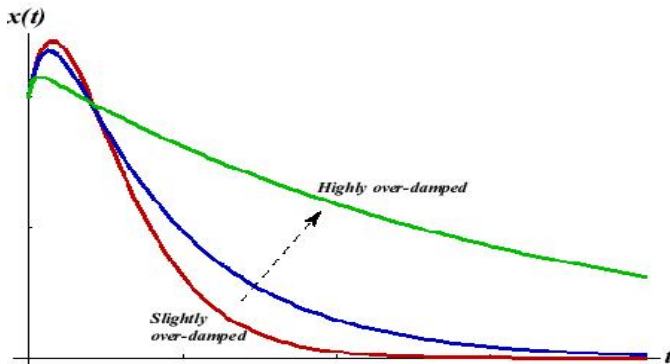


Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t} (c_1 + c_2 t)$$

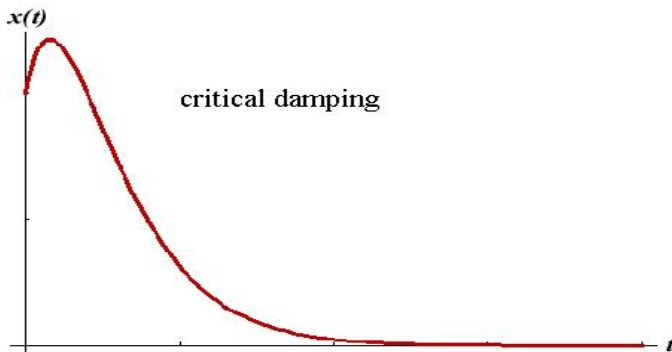


Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

$$x(t) = e^{-\lambda t} (c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

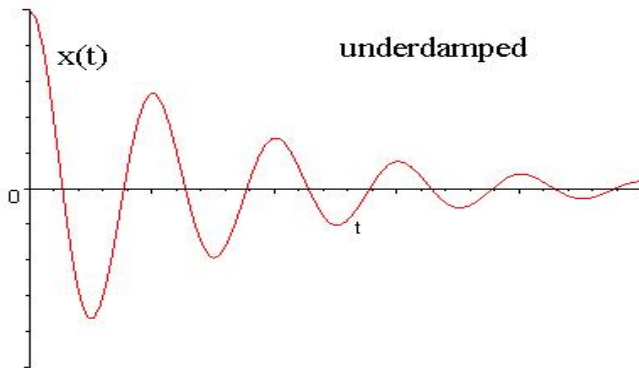


Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Comparison of Damping

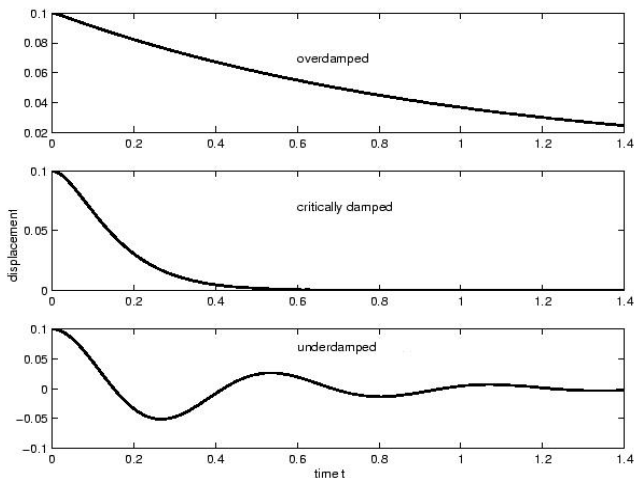


Figure: Comparison of three damping types.

Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

$$m = 2 \text{ kg}, \quad \beta = 10, \quad k = 12 \text{ N/m}$$

$$2x'' + 10x' + 12x = 0 \Rightarrow x'' + 5x' + 6x = 0$$

$$2\lambda = 5 \Rightarrow \lambda = \frac{5}{2}, \quad \omega^2 = 6$$

$$\lambda^2 - \omega^2 = \left(\frac{5}{2}\right)^2 - 6 = \frac{25}{4} - \frac{24}{4} = \frac{1}{4} > 0$$

This system is over damped.

The characteristic equation is

$$r^2 + 5r + 6 = 0$$

$$(r+2)(r+3) = 0 \Rightarrow$$

$$r_1 = -2$$

$$r_2 = -3$$

two
distinct
real roots
as expected.

Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

$$m=3, \quad \beta=12, \quad k=12 \quad \Rightarrow \quad 3x'' + 12x' + 12x = 0$$

$$x'' + 4x' + 4x = 0 \quad \text{so} \quad 2\lambda = 4 \quad \text{and} \quad \omega^2 = 4$$

$$\lambda^2 - \omega^2 = 2^2 - 4 = 0 \Rightarrow \quad \text{The system is critically damped.}$$

$$x'' + 4x' + 4x = 0, \quad x(0) = 0, \quad x'(0) = -1$$

$$r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0 \Rightarrow r = -2$$

General solution

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

$$x'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

$$x(0) = c_1 e^0 + c_2 \cdot 0 e^0 = 0 \Rightarrow c_1 = 0$$

$$x'(0) = 0 + c_2 e^0 - 2c_2 \cdot 0 e^0 = -1 \Rightarrow c_2 = -1$$

The equation of motion is

$$x(t) = -te^{-2t}$$

5.1.3: Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m \frac{d^2 x}{dt^2} = -\beta \frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2 x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = F(t)$$

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$.
Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and} \quad (2) \quad \gamma = \omega.$$

The DE is

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

with complementary solution

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$$x_p = A \cos(\gamma t) + B \sin(\gamma t)$$

If $\gamma \neq \omega$, this doesn't duplicate x_c so this guess works.

If $\gamma = \omega$, this duplicates x_c . So

$$x_p = (A \cos(\gamma t) + B \sin(\gamma t)) t$$

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$.
Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and} \quad (2) \quad \gamma = \omega.$$

$$\text{Case (1): } x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2} \left(\sin(\gamma t) - \frac{\gamma}{\omega} \sin(\omega t) \right)$$

If $\gamma \approx \omega$, the amplitude of motion could be rather large!

Pure Resonance

Case (2): $x'' + \omega^2 x = F_0 \sin(\omega t)$, $x(0) = 0$, $x'(0) = 0$

$$x(t) = \frac{F_0}{2\omega^2} \sin(\omega t) - \frac{F_0}{2\omega} t \cos(\omega t)$$

Note that the amplitude, α , of the second term is a function of t :

$$\alpha(t) = \frac{F_0 t}{2\omega}$$

which grows without bound!

► Forced Motion and Resonance Applet

Choose "Elongation diagram" to see a plot of displacement. Try exciter frequencies close to ω .