October 14 Math 2306 sec. 53 Fall 2019

Section 10: Variation of Parameters

We are still considering nonhomogeneous, linear ODEs. Consider equations of the form

$$y'' + y = \tan x, \quad \text{or} \quad x^2 y'' + xy' - 4y = e^x.$$

The method of undetermined coefficients is not applicable to either of these. We require another approach.

Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

$y_c = c_1 y_1(x) + c_2 y_2(x)$

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

We have two unknowns but only one equation (the ODE), so we'll introduce a 2nd equation.

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2' + \underbrace{u_1' y_1 + u_2' y_2}_{\text{set this to zero}}$$

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

Our 2nd equation is $u_1' y_1 + u_2' y_2 = 0$.

Substitute

$$y_p'' + P(x) y_p' + Q(x) y_p = g$$

$$u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' + P(x)\left(u_1 y_1' + u_2 y_2'\right) + Q(x)\left(u_1 y_1 + u_2 y_2\right) = g$$

Collect by u_1, u_2, u_1' and u_2'.

$$u_1' y_1' + u_2' y_2' + u_1 \underbrace{\left(y_1'' + P(x) y_1' + Q(x) y_1\right)}_{=0} + u_2 \underbrace{\left(y_2'' + P(x) y_2' + Q(x) y_2\right)}_{=0} = g$$

Because y_1 and y_2 solve $y_i'' + P(x)y_i' + Q(x)y_i = 0$

This reduces to $u_1'y_1' + u_2'y_2' = g$

The u's satisfy the system

$$u_1'y_1 + u_2'y_2 = 0$$

$$u_1'y_1' + u_2'y_2' = g$$

As a matrix equation, this is

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

For the system $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$

If $D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, then

$$x_1 = \frac{D_1}{D} \quad \text{and} \quad x_2 = \frac{D_2}{D}$$

where $D_1 = \begin{vmatrix} e & b \\ f & d \end{vmatrix}$ and $D_2 = \begin{vmatrix} a & e \\ c & f \end{vmatrix}$

This is Cramer's rule.

We have $u_1' = \dfrac{W_1}{W}$ and $u_2' = \dfrac{W_2}{W}$

where

$$W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

$$W_1 = \det \begin{bmatrix} 0 & y_2 \\ g & y_2' \end{bmatrix} = 0 - g y_2 = -g y_2$$

$$W_2 = \det \begin{bmatrix} y_1 & 0 \\ y_1' & g \end{bmatrix} = y_1 g - 0 = g y_1$$

So $$u_1 = \int \frac{-g y_2}{W}\, dx \quad \text{and} \quad u_2 = \int \frac{g y_1}{W}\, dx$$

and $$y_p = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$$