October 14 Math 2306 sec 54 Fall 2015

Section 5.1.1: Free Undamped Spring/Mass Systems

Simple Harmonic Motion: for displacement $x(t)$ of the mass

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

called the **equation of motion.** Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$
- the circular frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0\cos(\omega t) + \frac{x_1}{\omega}\sin(\omega t) = A\sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by (for a sine representation)

$$\sin\phi = \frac{x_0}{A}, \quad \text{with} \quad \cos\phi = \frac{x_1}{\omega A}.$$

5.1.2: Free Damped Motion

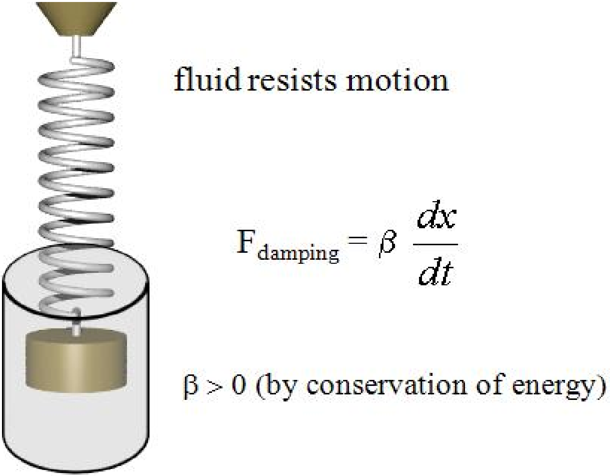

Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

$$\text{Force} = \text{Force of spring} + \text{Force of damping}$$

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$

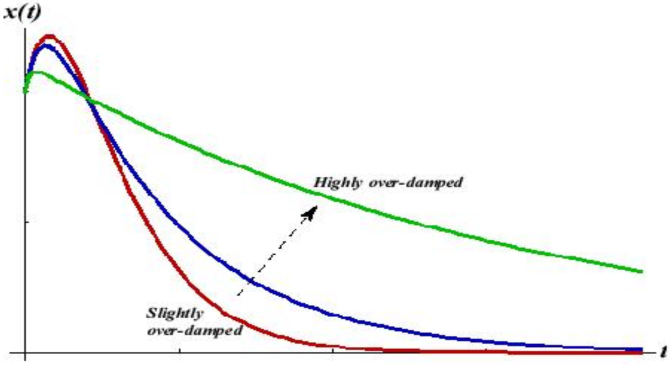

Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

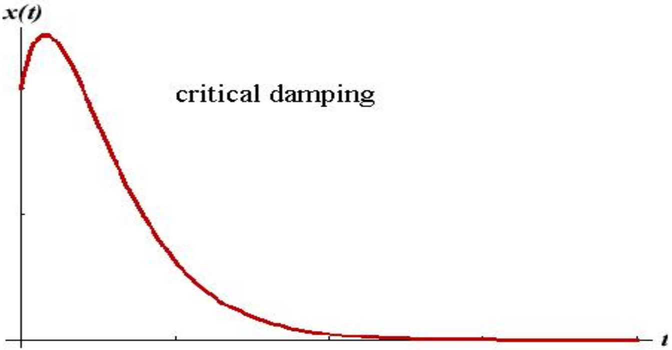

Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

$$x(t) = e^{-\lambda t}\left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

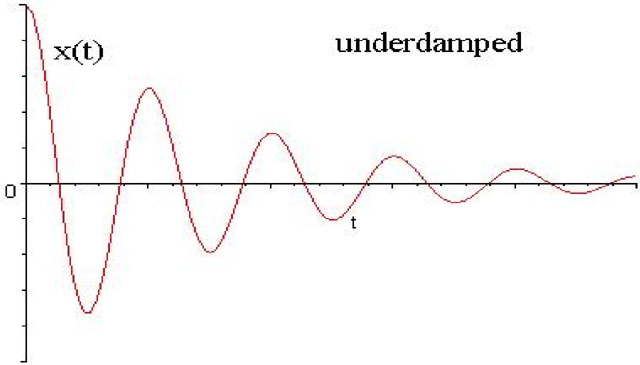

Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Comparison of Damping

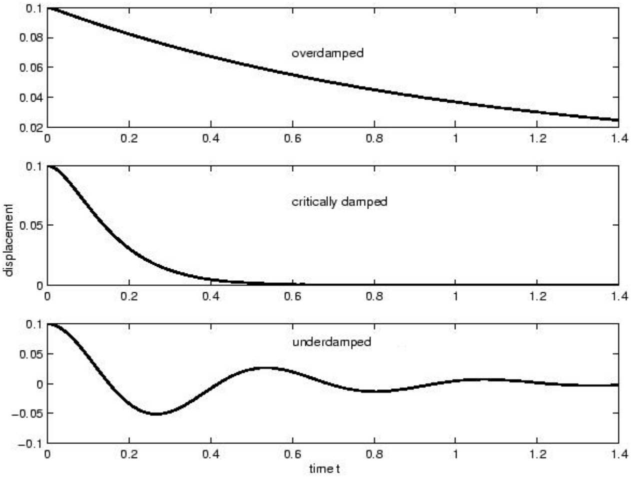

Figure: Comparison of three damping types.

Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

$$m = 2\text{ kg}, \quad \beta = 10\ \frac{\text{kg}}{\text{sec}}, \quad k = 12\ \frac{\text{N}}{\text{m}}$$

$$mx'' + \beta x' + kx = 0 \implies 2x'' + 10x' + 12x = 0$$

$$x'' + 5x' + 6x = 0 \implies 2\lambda = 5 \quad \text{and} \quad \omega^2 = 6$$

$$\lambda^2 - \omega^2 = \left(\frac{5}{2}\right)^2 - 6 = \frac{25}{4} - \frac{24}{4} = \frac{1}{4} > 0$$

The system is overdamped.

$x'' + 5x' + 6x = 0$ has characteristic eqn

$r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0 \implies$ $r_1 = -2$, $r_2 = -3$

2 distinct real roots

as expected

Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

$$m = 3\ \text{kg}, \quad \beta = 12\ \frac{\text{kg}}{\text{sec}}, \quad k = 12\ \text{N/m}$$

$$mx'' + \beta x' + kx = 0, \quad 3x'' + 12x' + 12x = 0$$

$$x'' + 4x' + 4x = 0 \qquad 2\lambda = 4, \quad \omega^2 = 4$$

$$\lambda^2 - \omega^2 = 4 - 4 = 0 \implies \text{critically damped}$$

$$x'' + 4x' + 4x = 0, \quad x(0) = 0, \quad x'(0) = -1$$

$$r^2 + 4r + 4 = 0 \implies (r+2)^2 = 0 \implies r = -2 \text{ repeated}$$

$$x(t) = C_1 e^{-2t} + C_2 t e^{-2t}$$

$$x'(t) = -2C_1 e^{-2t} + C_2 e^{-2t} - 2C_2 t e^{-2t}$$

$$x(0) = C_1 e^0 + C_2 \cdot 0 e^0 = 0 \implies C_1 = 0$$

$$x'(0) = 0 + C_2 e^0 - 2C_2 \cdot 0 e^0 = -1 \implies C_2 = -1$$

The equation of motion is

$$x(t) = -t e^{-2t}.$$

5.1.3: Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

$2\lambda = \frac{\beta}{m}$ and $\omega^2 = \frac{k}{m}$

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$. Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and (2)} \quad \gamma = \omega.$$

The DE is

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

with complementary solution

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$

If $\gamma \neq \omega$, this doesn't duplicate x_c. So it's a correct form.

If $\gamma = \omega$, this duplicates x_c. The correct form is

$$x_p = (A\cos(\gamma t) + B\sin(\gamma t))t = At\cos(\omega t) + Bt\sin(\omega t)$$

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$. Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and (2)} \quad \gamma = \omega.$$

$$\text{Case (1):} \quad x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2} \left(\sin(\gamma t) - \frac{\gamma}{\omega} \sin(\omega t) \right)$$

If $\gamma \approx \omega$, the amplitude of motion could be rather large!

Pure Resonance

$$\text{Case (2):} \quad x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{2\omega^2}\sin(\omega t) - \frac{F_0}{2\omega} t \cos(\omega t)$$

Note that the amplitude, α, of the second term is a function of t:

$$\alpha(t) = \frac{F_0 t}{2\omega}$$

which grows without bound!

Forced Motion and Resonance Applet

Choose "Elongation diagram" to see a plot of displacement. Try exciter frequencies close to ω.

5.1.4: Series Circuit Analog

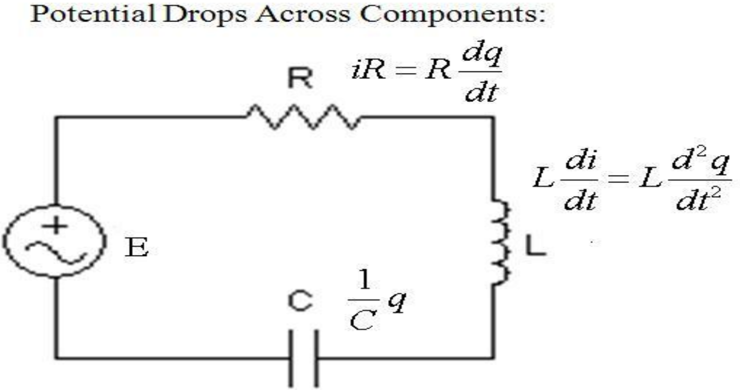

Figure: Kirchhoff's Law: The charge q on the capacitor satisfies $Lq'' + Rq' + \frac{1}{C}q = E(t)$.

LRC Series Circuit (Free Electrical Vibrations)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

If the applied force $E(t) = 0$, then the **electrical vibrations** of the circuit are said to be **free.** These are categorized as

$$\begin{array}{llcl} \textbf{overdamped} \text{ if} & R^2 - 4L/C & > & 0, \\ \textbf{critically damped} \text{ if} & R^2 - 4L/C & = & 0, \\ \textbf{underdamped} \text{ if} & R^2 - 4L/C & < & 0. \end{array}$$

Example

An *LRC* series circuit with no applied force has an inductance of $L = 2\text{h}$ and capacitance of $C = 5 \times 10^{-3}\text{f}$. Determine the condition on the resistor such that the electrical vibrations are

(a) Overdamped,

(b) Critically damped, or

(c) Underdamped.