October 15 MATH 1113 sec. 51 Fall 2018

Section 5.5: Solving Exponential and Logarithmic Equations

Base-Exponent Equality For any $a > 0$ with $a \neq 1$, and for any real numbers x and y

$$a^x = a^y \quad \text{if and only if} \quad x = y.$$

Logarithm Equality For and $a > 0$ with $a \neq 1$, and for any positive numbers x and y

$$\log_a x = \log_a y \quad \text{if and only if} \quad x = y.$$

Inverse Function For any $a > 0$ with $a \neq 1$

$$a^{\log_a x} = x \quad \text{for every} \quad x > 0$$

$$\log_a(a^x) = x \quad \text{for every real} \quad x.$$

Example

Solve the equation $3^{2x+1} = 81$. Show that the same result is obtained by two approaches.

(a) by using the fact that $81 = 3^4$ and equating exponents.

$3^{2x+1} = 81 \Rightarrow 3^{2x+1} = 3^4$

$a^x = a^y \Rightarrow x = y$

$\Rightarrow 2x+1 = 4$

$2x = 3$

$x = \frac{3}{2}$

Example

Solve the equation $3^{2x+1} = 81$. Show that the same result is obtained by two approaches.

(b) by using the base 3 logarithm as an inverse function.

$$3^{2x+1} = 81$$ Take log base 3

$$\log_3\left(3^{2x+1}\right) = \log_3(81)$$ $81 = 3^4$

$$(2x+1)\log_3(3) = 4$$

$$(2x+1)\cdot 1 = 4 \Rightarrow 2x = 3$$

$$x = \frac{3}{2}$$

Example

Find an exact solution[1] to the equation

$2^{x+1} = 5^x$

we can use any base log. We'll use the natural log. Take ln of both side

$$\ln\left(2^{x+1}\right) = \ln\left(5^x\right) \Rightarrow (x+1)\ln 2 = x\ln 5$$

Solve for x

$$x\ln 2 + \ln 2 = x\ln 5$$

$$\ln 2 = x\ln 5 - x\ln 2 \Rightarrow x(\ln 5 - \ln 2) = \ln 2$$

$$x = \frac{\ln 2}{\ln 5 - \ln 2} = \frac{\ln 2}{\ln\left(\frac{5}{2}\right)}$$

[1] An exact solution may be a number such as $\sqrt{2}$ or $\ln(7)$ which requires a calculator to approximate as a decimal.

Graphical Solution to $2^{x+1} = 5^x$

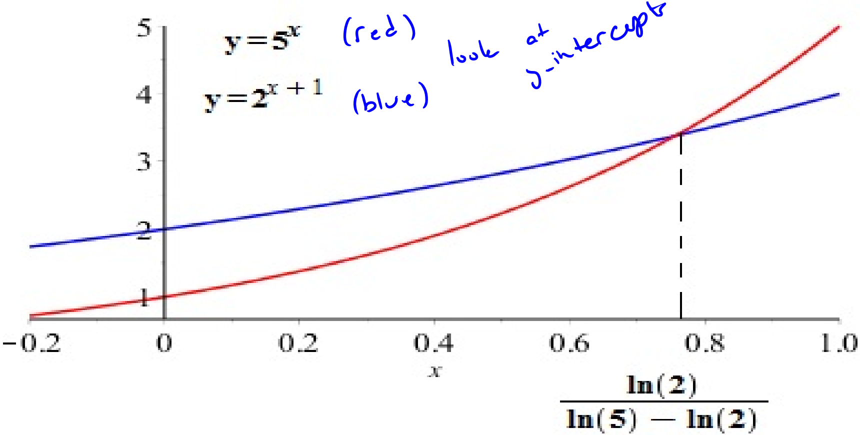

Figure: Plots of $y = 2^{x+1}$ and $y = 5^x$ together. The curves intersect at the solution $x = \ln 2/(\ln 5 - \ln 2) \approx 0.7565$.

Which curve is $y = 2^{x+1}$, red or blue?

Question

An exact solution to $3^{-x} = 4^{x-1}$ can be found using the natural logarithm. An exact solution is

$\ln(3^{-x}) = \ln(4^{x-1})$

$-x \ln 3 = (x-1)\ln 4$

$\ln 4 = x\ln 4 + x\ln 3$

$= x(\ln 4 + \ln 3)$

(a) $x = \dfrac{\ln 4}{\ln 4 - \ln 3}$

(b) $x = \dfrac{\ln 3}{\ln 4 - \ln 3}$

(c) $x = \dfrac{\ln 4}{\ln 4 + \ln 3}$

(d) $x = \dfrac{\ln 3}{\ln 4 + \ln 3}$

(e) I know how to do this, but my answer is not here

An Observation

To solve $2^{x+1} = 5^x$, we used the natural log. But we have choices. Use the change of base formula to show that our solution

$$\frac{\ln 2}{\ln 5 - \ln 2} = \frac{\log 2}{\log 5 - \log 2}$$

We know that $\ln 2 = \dfrac{\log 2}{\log e}$, $\ln 5 = \dfrac{\log 5}{\log e}$

$$\frac{\ln 2}{\ln 5 - \ln 2} = \frac{\frac{\log 2}{\log e}}{\frac{\log 5}{\log e} - \frac{\log 2}{\log e}} \cdot \frac{\log e}{\log e} = \frac{\log 2}{\log 5 - \log 2}$$

Question

Jack and Diane are solving $3^{-x} = 4^{x-1}$. They arrive at the solutions

$$\text{Jack's} \quad \frac{\ln 4}{\ln 4 + \ln 3} \qquad\qquad \text{Diane's} \quad \frac{\log_3(4)}{\log_3(4) + 1}.$$

$1 = \log_3(3)$

Which of the following statements is true?

(a) Jack's answer is correct, and Diane's is incorrect.

(b) Diane's answer is correct, and Jack's is incorrect.

(c) Both answers are correct; they are the same number.

(d) Both answers are incorrect.

Log Equations & Verifying Answers

Double checking answers is always recommended. When dealing with functions whose domains are restricted, **answer verification** is critical.

Use properties of logarithms to solve the equation
$\log(x-1)+\log(x-2)=\log 12$

Sum property of logs

$\log((x-1)(x-2)) = \log 12$

$\log a = \log b \Rightarrow a = b$

$\log(x^2-3x+2) = \log 12$

$x^2-3x+2 = 12$

solve this quadratic

$x^2-3x-10 = 0$

$$(x-5)(x+2)=0$$

$x=5$ and $x=-2$ both solve the quadratic.

Check answers in the original equation.

$$\log(x-1)+\log(x-2)=\log 12$$

plug in 5 $\quad \log(5-1)+\log(5-2)$ $\quad$ 5 solves it.

$$=\log 4+\log 3$$

$$=\log(4\cdot 3)=\log 12$$

plug in -2 $\quad \log(-2-1)+\log(-2-2)$

$$\log(-3)+\log(-4)$$ Undefined!

The only solution is 5.