October 15 MATH 1113 sec. 52 Fall 2018

Section 5.5: Solving Exponential and Logarithmic Equations

Base-Exponent Equality For any $a > 0$ with $a \neq 1$, and for any real numbers x and y

$$a^x = a^y \quad \text{if and only if} \quad x = y.$$

Logarithm Equality For and $a > 0$ with $a \neq 1$, and for any positive numbers x and y

$$\log_a x = \log_a y \quad \text{if and only if} \quad x = y.$$

Inverse Function For any $a > 0$ with $a \neq 1$

$$a^{\log_a x} = x \quad \text{for every} \quad x > 0$$

$$\log_a(a^x) = x \quad \text{for every real} \quad x.$$

Example

Solve the equation $3^{2x+1} = 81$. Show that the same result is obtained by two approaches.

(a) by using the fact that $81 = 3^4$ and equating exponents.

$$3^{2x+1} = 81 \Rightarrow 3^{2x+1} = 3^4$$

Since $a^x = a^y$

$\Rightarrow x = y$

$$2x+1 = 4$$

$$2x = 3$$

$$x = \frac{3}{2}$$

Example

Solve the equation $3^{2x+1} = 81$. Show that the same result is obtained by two approaches.

(b) by using the base 3 logarithm as an inverse function.

$3^{2x+1} = 81$ Take log base 3 of both sides.

$\log_3(3^{2x+1}) = \log_3(81)$ since $3^4 = 81$

$(2x+1)\log_3(3) = 4$

$(2x+1)\cdot 1 = 4 \Rightarrow 2x = 3 \Rightarrow x = \frac{3}{2}$

Example

Find an exact solution[1] to the equation

$2^{x+1} = 5^x$

We'll use the natural log.

$\ln\left(2^{x+1}\right) = \ln 5^x$

$(x+1)\ln 2 = x\ln 5$

solve for x

$x\ln 2 + \ln 2 = x\ln 5 \Rightarrow \ln 2 = x\ln 5 - x\ln 2$

$x(\ln 5 - \ln 2) = \ln 2$

$x = \dfrac{\ln 2}{\ln 5 - \ln 2} = \dfrac{\ln 2}{\ln\left(\frac{5}{2}\right)}$

[1] An exact solution may be a number such as $\sqrt{2}$ or $\ln(7)$ which requires a calculator to approximate as a decimal.

Graphical Solution to $2^{x+1} = 5^x$

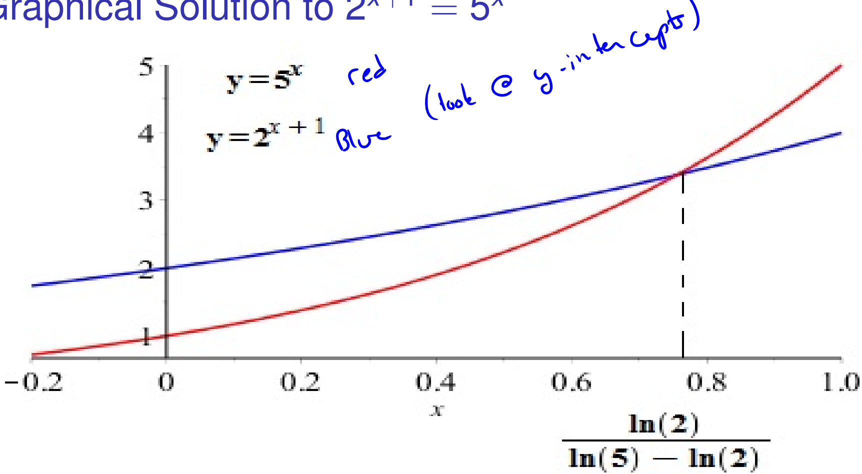

Figure: Plots of $y = 2^{x+1}$ and $y = 5^x$ together. The curves intersect at the solution $x = \ln 2/(\ln 5 - \ln 2) \approx 0.7565$.

Which curve is $y = 2^{x+1}$, red or blue?

Question

An exact solution to $3^{-x} = 4^{x-1}$ can be found using the natural logarithm. An exact solution is

(a) $x = \dfrac{\ln 4}{\ln 4 - \ln 3}$

(b) $x = \dfrac{\ln 3}{\ln 4 - \ln 3}$

(c) $x = \dfrac{\ln 4}{\ln 4 + \ln 3} = \dfrac{-\ln 4}{-\ln 4 - \ln 3}$

(d) $x = \dfrac{\ln 3}{\ln 4 + \ln 3}$

(e) I know how to do this, but my answer is not here

$$\ln 3^{-x} = \ln 4^{x-1}$$

$$-x \ln 3 = (x-1)\ln 4$$

$$= x\ln 4 - \ln 4$$

$$\ln 4 = x\ln 4 + x\ln 3$$

$$= x(\ln 4 + \ln 3)$$

$$x = \frac{\ln 4}{\ln 4 + \ln 3}$$

An Observation

To solve $2^{x+1} = 5^x$, we used the natural log. But we have choices. Use the change of base formula to show that our solution

$$\frac{\ln 2}{\ln 5 - \ln 2} = \frac{\log 2}{\log 5 - \log 2}$$

By the change of base formula

$$\ln 2 = \frac{\log 2}{\log e} \qquad \ln 5 = \frac{\log 5}{\log e}$$

$$\frac{\ln 2}{\ln 5 - \ln 2} = \frac{\frac{\log 2}{\log e}}{\frac{\log 5}{\log e} - \frac{\log 2}{\log e}} \cdot \left(\frac{\log e}{\log e}\right)$$

$$= \frac{\log 2}{\log 5 - \log 2}$$

Question

Jack and Diane are solving $3^{-x} = 4^{x-1}$. They arrive at the solutions

$$\text{Jack's} \quad \frac{\ln 4}{\ln 4 + \ln 3} \qquad\qquad \text{Diane's} \quad \frac{\log_3(4)}{\log_3(4) + 1}.$$

$\log_3(3) = 1$

Which of the following statements is true?

(a) Jack's answer is correct, and Diane's is incorrect.

(b) Diane's answer is correct, and Jack's is incorrect.

(c) Both answers are correct; they are the same number.

(d) Both answers are incorrect.

Log Equations & Verifying Answers

Double checking answers is always recommended. When dealing with functions whose domains are restricted, **answer verification** is critical.

Use properties of logarithms to solve the equation
$\log(x-1) + \log(x-2) = \log 12$

$$\log\left((x-1)(x-2)\right) = \log 12$$

$$\log a = \log b \Rightarrow a = b$$

$$\log(x^2-3x+2) = \log 12$$

$$x^2-3x+2 = 12$$

$$x^2-3x-10 = 0$$

$$(x+2)(x-5) = 0 \Rightarrow x = 5 \text{ or } x = -2$$

We have to determine if both answers solve the original equation.

$$\log(x-1) + \log(x-2) = \log 12$$

Check 5: $\log(5-1) + \log(5-2) =$

$\log(4) + \log(3) = \log(4\cdot 3) = \log 12$ ✓

5 solves the equation

Check -2: $\log(-2-1) + \log(-2-2) =$

$\log(-3) + \log(-4)$ both undefined

-2 is not a solution