

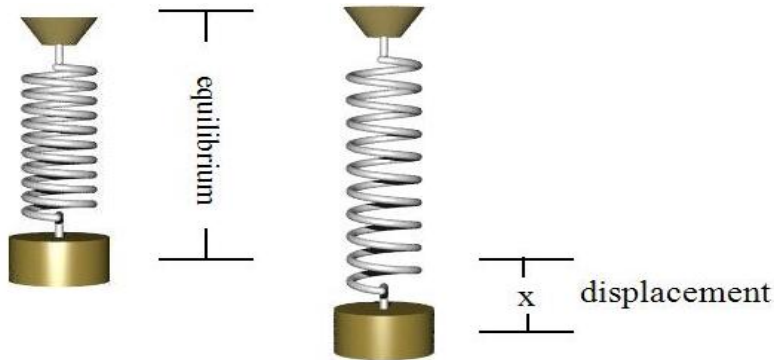
Section 11: Linear Mechanical Equations

Simple Harmonic Motion

We consider a flexible spring from which a mass is suspended. In the absence of any damping forces (e.g. friction, a dash pot, etc.), and free of any external driving forces, any initial displacement or velocity imparted will result in **free, undamped motion**—a.k.a. **simple harmonic motion**.

▶ Harmonic Motion gif

Building an Equation: Hooke's Law



At equilibrium, displacement $x(t) = 0$.

$$\text{Hooke's Law: } F_{\text{spring}} = k x$$

Figure: In the absence of any displacement, the system is at equilibrium. Displacement $x(t)$ is measured from equilibrium $x = 0$.

Building an Equation: Hooke's Law

Newton's Second Law: $F = ma$ Force = mass times acceleration

$$a = \frac{d^2x}{dt^2} \implies F = m \frac{d^2x}{dt^2}$$

Hooke's Law: $F = kx$ Force exerted by the spring is proportional to displacement

The force imparted by the spring opposes the direction of motion.

$$mx'' + kx = 0 \implies x'' + \frac{k}{m}x = 0$$

$$m \frac{d^2x}{dt^2} = -kx \implies x'' + \omega^2 x = 0 \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}$$

Convention We'll Use: Up will be positive ($x > 0$), and down will be negative ($x < 0$). This orientation is arbitrary and follows the convention in Trench.


Obtaining the Spring Constant (US Customary Units)

If an object with weight W pounds stretches a spring δx feet¹ from its length with no mass attached, then by Hooke's law we compute the spring constant via the equation

$$W = k\delta x.$$

The units for k in this system of measure are lb/ft.

$$k = \frac{W}{\delta x} \frac{\text{lb}}{\text{ft}}$$

¹Note that $\delta x = w/\text{mass equilibrium} - w/\text{no mass equilibrium}$. 

Obtaining the Spring Constant (US Customary Units)

Note also that Weight = mass \times acceleration due to gravity. Hence if we know the weight of an object, we can obtain the mass via

$$W = mg.$$

We typically take the approximation $g = 32 \text{ ft/sec}^2$. The units for mass are $\text{lb sec}^2/\text{ft}$ which are called slugs.

$$\text{mass} \quad m = \frac{W}{g} \frac{\text{lb}}{\text{ft}/\text{sec}^2} = \frac{W}{g} \frac{\text{lb sec}^2}{\text{ft}}$$

\uparrow
slugs

Obtaining the Spring Constant (SI Units)

In SI units, the weight would be expressed in Newtons (N). The appropriate units for displacement would be meters (m). In these units, the spring constant would have units of N/m.

It is customary to describe an object by its mass in kilograms. When we encounter such a description, we deduce the weight in Newtons

$$W = mg \quad \text{taking the approximation} \quad g = 9.8 \text{ m/sec}^2.$$

Obtaining the Spring Constant: *Displacement in Equilibrium*

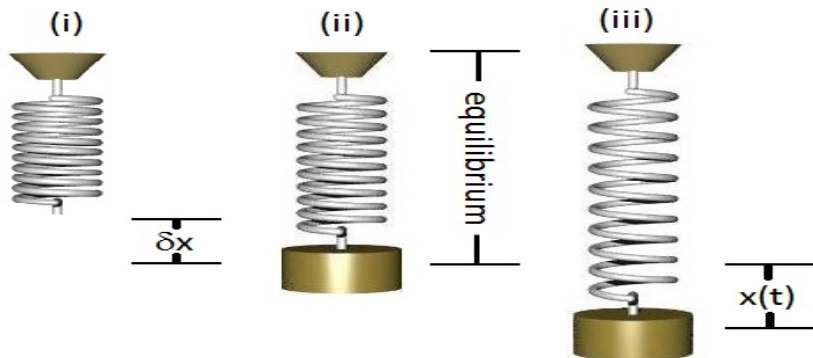


Figure: (i) Spring only *equilibrium*. (ii) Spring-mass system **equilibrium**. The difference δx will be called *displacement in equilibrium*. Our variable $x(t)$ will be displacement of the **Spring-Mass system**.

Obtaining the Spring Constant: *Displacement in Equilibrium*

If an object stretches a spring δx units from its length (with no object attached), we may say that it stretches the spring δx units *in equilibrium*. Applying Hooke's law with the weight as force, we have

$$\text{Weight} \quad W = mg = k\delta x. = W \quad \frac{mg}{m\delta x} = \frac{k\delta x}{m\delta x}$$

We observe that the value ω can be deduced from δx by

$$\omega^2 = \frac{k}{m} = \frac{g}{\delta x}.$$

Provided that values for δx and g are used in appropriate units, ω is in units of per second.

Simple Harmonic Motion

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \quad (1)$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

called the **equation of motion**.

Caution: The phrase *equation of motion* is used differently by different authors. Some, including Trench, use this phrase to refer the ODE of which (1) would be the example here. Others use it to refer to the **solution** to the associated IVP.

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

Characteristics of the system include

- ▶ the period $T = \frac{2\pi}{\omega}$,
- ▶ the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$ ²
- ▶ the circular (or angular) frequency ω , and
- ▶ the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

²Various authors call f the natural frequency and others use this term for ω . 

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by

$$\sin \phi = \frac{x_0}{A}, \quad \text{with} \quad \cos \phi = \frac{x_1}{\omega A}.$$

(Alternatively, we can let $x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \cos(\omega t - \hat{\phi})$ in which case $\hat{\phi}$ is defined by

$$\cos \hat{\phi} = \frac{x_0}{A}, \quad \text{with} \quad \sin \hat{\phi} = \frac{x_1}{\omega A}.$$

The phase shift defined above $\phi = \frac{\pi}{2} - \hat{\phi}$.)

Example

An object stretches a spring 6 inches in equilibrium. Assuming no driving force and no damping, set up the differential equation describing this system.

The ODE looks like $x'' + \omega^2 x = 0$

To find ω^2 we can use $\omega^2 = \frac{g}{\delta x}$

We're given $\delta x = 6$ in, so $g = 32$ ft/sec².

Converting δx to feet $\delta x = \frac{1}{2}$ ft

$$\omega^2 = \frac{32 \text{ ft/sec}^2}{\frac{1}{2} \text{ ft}} = 64 \frac{1}{\text{sec}^2}$$

The oDE is

$$x'' + 64x = 0$$

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet above equilibrium with an initial downward velocity of 24 ft/sec. Find the equation of motion, the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32 \text{ ft/sec}^2$.)

Let's find m and k and confirm that $\omega^2 = 64$.

Get mass: $W = mg$ here $W = 4 \text{ lb}$, $g = 32 \text{ ft/sec}^2$

$$m = \frac{W}{g} = \frac{4 \text{ lb}}{32 \text{ ft/sec}^2} = \frac{1}{8} \text{ slugs}$$

Get k : $W = k \Delta x$ here $W = 4 \text{ lb}$ and $\Delta x = \frac{1}{2} \text{ ft}$

$$k = \frac{W}{\Delta x} = \frac{4 \text{ lb}}{\frac{1}{2} \text{ ft}} = 8 \frac{\text{lb}}{\text{ft}}$$

$$\text{Then } \omega^2 = \frac{k}{m} = \frac{8 \frac{\text{lb}}{\text{ft}}}{\frac{1}{8} \text{ slug}} = 64 \frac{1}{\text{sec}^2}$$

The IVP is

$$x'' + 64x = 0$$

$$x(0) = 4 \quad \text{and} \quad x'(0) = -24$$

← positive
above
equilibrium

← negative
downward

The characteristic eqn is (use r instead of m)

$$r^2 + 64 = 0 \Rightarrow r^2 = -64$$

$$r = \pm \sqrt{-64} = 0 \pm 8i$$

$$\alpha = 0 \quad \beta = 8$$

$$x_1(t) = \cos(8t) \quad \text{and} \quad x_2(t) = \sin(8t)$$

$$x = c_1 \cos(8t) + c_2 \sin(8t)$$

Apply $x(0) = 4$ $x'(0) = -24$

$$x'(t) = -8c_1 \sin(8t) + 8c_2 \cos(8t)$$

$$x(0) = c_1 \cos(0) + c_2 \sin(0) = 4$$

$$c_1 = 4 \quad \Rightarrow \quad c_1 = 4$$

$$x'(0) = -8c_1 \sin(0) + 8c_2 \cos(0) = -24$$

$$8c_2 = -24 \quad \Rightarrow \quad c_2 = -3$$

The equation of motion is

$$x(t) = 4 \cos(8t) - 3 \sin(8t)$$

Period $T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$

Frequency $f = \frac{1}{T} = \frac{4}{\pi}$

Amplitude $A = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$

The phase shift ϕ satisfies

$$\sin \phi = \frac{x_0}{A} = \frac{4}{5} \quad \text{and} \quad \cos \phi = \frac{x_1/\omega}{A} = \frac{-3}{5}$$

Since $\sin \phi > 0$ and $\cos \phi < 0$

$$\frac{\pi}{2} < \phi < \pi \quad (\text{the smallest positive } \phi)$$

Use arc cosine

$$\phi = \cos^{-1}\left(\frac{-3}{5}\right) \approx 2.21$$

$$\phi \approx 126.86^\circ$$

Note

$$\begin{aligned}x &= x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) \\ &= A \sin(\omega t + \phi)\end{aligned}$$

$$\text{Let } A = \sqrt{x_0^2 + \left(\frac{x_1}{\omega}\right)^2}$$

$$x = A \left(\frac{x_0}{A} \cos(\omega t) + \frac{x_1/\omega}{A} \sin(\omega t) \right)$$

$$\text{If } \sin \phi = \frac{x_0}{A} \text{ and } \cos \phi = \frac{x_1/\omega}{A}$$

$$= A (\sin \phi \cos(\omega t) + \cos \phi \sin(\omega t))$$

$$= A \sin(\phi + \omega t)$$