October 16 Math 2306 sec 51 Fall 2015

5.1.3: Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

Forced Undamped Motion and Resonance

Solve the IVP for the driven spring mass system. Assume that $\gamma \neq \omega$.

$$x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, x'(0) = 0$$

Initial velocity of zero = at rest

$x = x_c + x_p$ Find x_c: $x'' + \omega^2 x = 0$

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

Find x_p: Use Method of Undetermined Coeffs.

This will work since $\gamma \neq \omega$

$$x_p = A \sin(\gamma t) + B \cos(\gamma t)$$

$$x_p' = \gamma A \cos(\gamma t) - \gamma B \sin(\gamma t)$$

$$x_p'' = -\gamma^2 A \sin(\gamma t) - \gamma^2 B \cos(\gamma t) = -\gamma^2 x_p$$

$$x_p'' + \omega^2 x_p = F_0 \sin(\gamma t)$$

$$-\gamma^2 x_p + \omega^2 x_p = F_0 \sin(\gamma t)$$

$$(\omega^2 - \gamma^2)\, x_p = F_0 \sin(\gamma t)$$

$$(\omega^2 - \gamma^2)\left(A \sin(\gamma t) + B\cos(\gamma t)\right) = F_0 \sin(\gamma t) + 0 \cos(\gamma t)$$

$$B = 0$$

$$A = \frac{F_0}{\omega^2 - \gamma^2}$$

$$\Rightarrow \quad x_p = \frac{F_0}{\omega^2-\gamma^2}\sin(\gamma t)$$

The general solution to the DE is

$$x = c_1\cos(\omega t) + c_2\sin(\omega t) + \frac{F_0}{\omega^2-\gamma^2}\sin(\gamma t)$$

$x(0)=0$

$x'(0)=0$

$$x' = -\omega c_1\sin(\omega t) + \omega c_2\cos(\omega t) + \frac{\gamma F_0}{\omega^2-\gamma^2}\cos(\gamma t)$$

$$x(0) = c_1 = 0 \quad , \quad x'(0) = \omega c_2 + \frac{\gamma F_0}{\omega^2-\gamma^2} = 0$$

$$c_2 = \frac{-1}{\omega}\,\frac{\gamma F}{\omega^2-\gamma^2}$$

The solution to the IVP is

$$x = \frac{-1}{\omega}\,\frac{\gamma F_0}{\omega^2-\gamma^2}\sin(\omega t) + \frac{F_0}{\omega^2-\gamma^2}\sin(\gamma t).$$

Rearranged

$$x = \frac{F_0}{\omega(\omega^2-\gamma^2)}\left(\omega\sin(\gamma t) - \gamma\sin(\omega t)\right)$$

$$x(t) = \frac{F_0}{\omega(\omega^2 - \gamma^2)}\left(\omega\sin(\gamma t) - \gamma\sin(\omega t)\right)$$

Evaluate $\displaystyle\lim_{\gamma\to\omega} x(t)$ $\displaystyle = \lim_{\gamma\to\omega} \frac{F_0}{\omega(\omega^2-\gamma^2)}\left[\omega \sin(\gamma t) - \gamma \sin(\omega t)\right]$

$= \text{"}\frac{0}{0}\text{"}$ indeterminate form

use l'Hospital's rule

$$= \lim_{\gamma\to\omega} F_0 \frac{\frac{d}{d\gamma}\left[\omega \sin(\gamma t) - \gamma\sin(\omega t)\right]}{\frac{d}{d\gamma}\left(\omega(\omega^2-\gamma^2)\right)}$$

$$= \lim_{\gamma \to \omega} \frac{F_0 \left(\omega t \cos(\gamma t) - \sin(\omega t) \right)}{\omega(-2\gamma)}$$

$$= \frac{F_0 \left[\omega t \cos(\omega t) - \sin(\omega t) \right]}{-2\omega^2}$$

$$= \underbrace{-\frac{F_0}{2\omega} t \cos(\omega t)}_{\text{resonance term}} + \frac{F_0}{2\omega^2} \sin(\omega t)$$