

## 5.1.3: Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force  $f(t)$  is applied to the system. The ODE governing displacement becomes

$$m \frac{d^2x}{dt^2} = -\beta \frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out  $m$  and let  $F(t) = f(t)/m$  to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = F(t)$$

## Forced Undamped Motion and Resonance

Solve the IVP for the driven spring mass system. Assume that  $\gamma \neq \omega$ .

$$x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, x'(0) = 0$$

Initial velocity of zero = "at rest"

$$x = x_c + x_p \quad \text{Get } x_c: \quad x'' + \omega^2 x = 0$$

$$x_c = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

Find  $x_p$ : Undetermined Coefficients

$$x_p = A \sin(\gamma t) + B \cos(\gamma t)$$

$$x_p' = \gamma A \cos(\gamma t) - \gamma B \sin(\gamma t)$$

$$x_p'' = -\gamma^2 A \sin(\gamma t) - \gamma^2 B \cos(\gamma t) = -\gamma^2 x_p$$

$$x_p'' + \omega^2 x_p = F_0 \sin(\omega t)$$

$$-\gamma^2 x_p + \omega^2 x_p = F_0 \sin(\omega t) \Rightarrow (\omega^2 - \gamma^2) x_p = F_0 \sin(\omega t)$$

$$(\omega^2 - \gamma^2) [A \sin(\omega t) + \beta \cos(\omega t)] = F_0 \sin(\omega t) + 0 \cdot \cos(\omega t)$$

$$(\omega^2 - \gamma^2) A = F_0 \Rightarrow A = \frac{F_0}{\omega^2 - \gamma^2}, \beta = 0$$

$$(\omega^2 - \gamma^2) \beta = 0$$

The general solution to the ODE is

$$x = C_1 \cos(\omega t) + C_2 \sin(\omega t) + \frac{F_0}{\omega^2 - \gamma^2} \sin(\gamma t)$$

$$x(0) = 0$$

$$x'(0) = 0$$

$$x' = -\omega C_1 \sin(\omega t) + \omega C_2 \cos(\omega t) + \frac{\gamma F_0}{\omega^2 - \gamma^2} \cos(\gamma t)$$

$$x(0) = C_1 = 0$$

$$x'(0) = \omega C_2 + \frac{\gamma F_0}{\omega^2 - \gamma^2} = 0 \Rightarrow C_2 = -\frac{1}{\omega} \frac{\gamma F_0}{\omega^2 - \gamma^2}$$

The solution to the IVP is

$$x = \frac{-1}{\omega} \frac{\gamma F_0}{\omega^2 - \gamma^2} \sin(\omega t) + \frac{F_0}{\omega^2 - \gamma^2} \sin(\gamma t)$$

This can be rearranged as

$$x = \frac{F_0}{\omega(\omega^2 - \gamma^2)} \left[ \omega \sin(\gamma t) - \gamma \sin(\omega t) \right]$$

$$x(t) = \frac{F_0}{\omega(\omega^2 - \gamma^2)} (\omega \sin(\gamma t) - \gamma \sin(\omega t))$$

Evaluate  $\lim_{\gamma \rightarrow \omega} x(t) = \lim_{\gamma \rightarrow \omega} \frac{F_0 (\omega \sin(\gamma t) - \gamma \sin(\omega t))}{\omega (\omega^2 - \gamma^2)}$

$= \frac{0}{0}$  Indeterminate form  
use L'Hopital's rule

\* Note  
differentiation  
is w.r.t.  
respect to  
 $\gamma$

$$= \lim_{\gamma \rightarrow \omega} \frac{F_0 \frac{d}{d\gamma} (\omega \sin(\gamma t) - \gamma \sin(\omega t))}{\omega \frac{d}{d\gamma} (\omega^2 - \gamma^2)}$$

$$= \lim_{\gamma \rightarrow \omega} \frac{F_0 (\omega t \cos(\gamma t) - \sin(\gamma t))}{\omega (-2\gamma)}$$

$$= \frac{F_0 (\omega t \cos(\omega t) - \sin(\omega t))}{-\omega^2}$$

$$= -\frac{F_0 t}{\omega} \cos(\omega t) + \frac{F_0}{\omega^2} \sin(\omega t)$$

here's the resonance term